Mathematics Student Manifesto

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1 Introduction

This manifesto is targeted toward undergraduate mathematics students at institutions of higher education in the United States. It is a compilation of my opinions on what you, a student in a college mathematics course, should be doing in order to get the most out of your experience, both in terms of knowledge and skills learned, and also in terms of your own enjoyment of the subject. These opinions are based on my experiences as a student and as a college professor.

Some of the things I write below may seem obvious to you. That's OK. Some of the things I write below may seem outlandish to you. That's OK, too. I invite you to read this document through completely, adhere to the tenets for a semester (or longer), engage in as many of the good practices as you are able, and see how it goes.

2 Tenets of the Mathematics Student

First, you should know that it's OK to be confused. Encountering new ideas in mathematics tends to be confusing. Mathematics is filled with many definitions, theorems, propositions, lemmas, corollaries, and conjectures, all of which are typically described using some manner of mathematical notation and specific language. It is completely normal for these ideas, definitions, theorems, and notations to appear opaque and mysterious at first. However, there are some things you can do to demystify everything.

Tenet 1. Make up your own examples.

Most textbooks have some examples which try to illustrate the ideas in the area of mathematics you are studying. Occasionally, you may encounter a text (or a professor's lecture notes) which do not have any examples following a definition or theorem, or there are very few examples given and they don't seem to be very enlightening. The best thing you can do as a student is to make up your own examples and explore them.

As an example of what I mean, consider the definition below (which I just made up).

Definition. A function $f\colon \mathbb{R} \to \mathbb{R}$ is **purple** if $f(x+2) = f(x-3)$ for all $x \in \mathbb{R}$.

Now, it may be entirely clear to you what a purple function is just by reading the definition. Or perhaps the definition may seem a bit odd and confusing. In either case, making up an example will be a great help. If you feel you understand what purple functions are, then just invent a function which you believe is purple and test it against the definition. If you don't understand what it means to be purple, then just start with any old function and try to figure out what the definition means. In general, a good approach is to start with the simplest example you can think of, and work up from there. So, let's start with the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 0$. To see if this function is purple, you only need to test it against the definition of purple. In this case, you need to see whether or not the statement $f(x+2) = f(x-3)$ is true for all $x \in \mathbb{R}$ for this function. A quick computation should show you that $f(x+2) = 0$ and $f(x-3) = 0$ regardless of what x is, so this function is indeed purple!

Tenet 2. If your example works, try to figure out why.

The function $f(x) = 0$ is purple essentially because no matter what x is, the output of the function will be 0. So the relevant property here is that the function f is constant. This sort of reflection has led you to a proposition about purple functions. Maybe you've thought of this already:

Proposition. All constant functions are purple. That is, if f is constant, then f is purple.

Once you have a working example, it's a good idea to try more complicated examples, gradually increasing the complexity of what you're working with.

Tenet 3. Start with simple cases and gradually increase the complexity.

After constant functions, perhaps the next simplest function to think about is $g(x) = x$. Is this function purple? To investigate, try picking a value for x, again starting with something simple, like $x = 0$. For g to purple, we need for $g(0+2)$ to be equal to $g(0-3)$. You can check that $g(0+2) = g(2) = 2$ and that $g(0-3) = g(-3) = -3$. Since $2 \neq -3$, it must be that g is not purple.

Tenet 4. If your example doesn't work, think about why not.

Non-examples are just as valuable as examples, sometimes more so. So be sure to notice when you have something which doesn't satisfy a particular definition or the hypotheses of some theorem, since you have an equally good chance to learn from things that *do* work as you do from things that *don't* work. Both examples and non-examples will help you understand the definitions you are working with.

Continuing the example above, the function $g(x) = x$ is not purple since there is a value for x (namely $x = 0$) which does not satisfy the conditions set out in the definition. But what really happened here? Well, for g to be purple, we needed the value of g to be the same when you move 2 to the right of x as it is when you move 3 to the left of x, and this just doesn't happen for g when $x = 0$. Ah, but now this observation points you to the following: for a function f to be purple, the value 2 to the right of x needs to be equal to the value 3 to the left of x, regardless of what x is. So in some sense, f needs to repeat itself every five. If sometime in your mathematical past you have heard of periodic functions, then you might guess that the following is true.

Conjecture. Purple functions are periodic with period 5.

This conjecture is something you think is likely to be true based on the experience you have from looking at examples. You'll want to make conjectures often, and test them out equally as often.

Tenet 5. Make conjectures based on your observations. Investigate, refine, and extend your conjectures.

This is what mathematicians do, and it's how new mathematics gets created. This last point is very important to note. Because of this, you want to spend a lot of time working through examples and exercises.

Tenet 6. Work out a great variety of exercises.

In mathematics, like many other disciplines, understanding the vocabulary is critical. In mathematics, unlike many other disciplines, definitions are stated in precise, unambiguous terms. (Or at least as precise and unambiguous as is possible using language.) Because of the level of precision, it is absolutely essential for you to not merely memorize the definition, but internalize it. Definitions are carefully chosen to support the concepts they are trying to convey.

Tenet 7. Understand the vocabulary in the area of mathematics you are studying.

Creating examples and non-examples, like we did above, is typically the best way to understand a definition.

Statements in mathematics are quite dense; a lot of information is packed into few words. Because of this, it takes not only slow, careful reading to understand mathematical writing, but it also requires action on the part of you, the reader. You should get into the habit of writing everything out when you are reading mathematics. Keep a stock of paper and some pencils nearby and make heavy use of them while you are reading.

Tenet 8. Actively write while reading mathematics.

This writing can be more than merely copying the text. It can include creation and exploration of examples and non-examples, conjecturing, drawing pictures, and any other number of meditations.

There is, perhaps, a stereotype about mathematicians as being solitary, reclusive figures. Most mathematicians (this author included) increase their understanding the most when talking things through with others. The process of communicating ideas to others will typically help to clarify them for yourself. Do this often.

Tenet 9. Talk with your peers about mathematics.

You have to be willing to listen, as well. In mathematics, there is no room for ego. Examples, computations, and proofs are either correct, or else they are flawed. No matter how much you may want a statement to be true, or desperately need it to be false, in the end you have no control over its truth value. For this reason, you need to be critical of your own thinking, and that of others.

Tenet 10. Be critical of your work and of the work of others.

It is preferable to be kind with your criticisms, but essential to maintain honesty while doing so. Give yourself and others the chance to understand any misconceptions, and therefore the opportunity to correct them and learn from them.

Tenet 11. Learn from your mistakes.

When working on any particular exercise, you have complete control over the steps you take. Certain sequences of steps may be more efficient than others, but until you build up your intuition, there is no reason for you to know which route will be the best.

Tenet 12. Allow yourself to take the "wrong" path.

Above, you saw how non-examples can be as informative as examples. In the same way, taking a sequence of steps to try to solve a problem which do not seem to lead to a solution can be just as valuable as finding the solution right away. My favorite example of this involves integration by parts, a technique of integration covered in a standard second semester calculus course. With appropriate assumptions on functions u and v, integration by parts is stated as follows:

Theorem. $\int u\,dv = uv - \int v\,du.$

In applying this theorem, you want to find an anti-derivative of a particular function, something like $f(x) = x\sin(x)$. Viewing $f(x)$ as the product of the two functions $g(x) = x$ and $h(x) = \sin(x)$, you can choose either to integrate g and differentiate h, or instead integrate h and differentiate g. Either choice you make is fine, as far as integration by parts is concerned, and you'll be left with some new expression to integrate which depends on the choice you made. Typically, the goal is to end up with an expression which is simpler, or at least no more complicated, than the original. Usually, one choice is "better" than the other. But more often than not, there is something to be learned by taking the "wrong" path. And perhaps surprisingly, sometimes both paths lead to a solution.

This phenomenon is common in mathematics. Two seemingly different paths can lead to independent solutions of the same problem. So, whenever you have found one solution to a problem, you should start looking for others. Maybe a simpler, cleaner solution is waiting for you and is within your grasp. Or, just maybe, you'll find a new solution which no one has ever thought of before using techniques that you yourself have invented.

Tenet 13. Once you solve a problem, look for another solution using a different approach.

The last tenet is one to take quite seriously. In 1995, Andrew Wiles published a solution to Fermat's Last Theorem in a paper which exceeds 100 pages in length. Here is the theorem which Wiles proved, more than 350 years after Fermat conjectured it in 1637:

Theorem (Fermat's Last Theorem)**.** No three positive integers a, b, and c satisfy the equation $a^n + b^n = c^n$ for any integer n greater than 2.

One would hope for a simpler proof to such a brief, elegant statement. Perhaps other avenues of investigation will yield a shorter proof of this theorem.

3 Good Practices

There are some personal habits which you can adopt which will make your mathematics courses go more smoothly, will help you get the most out of your time, and will help you adhere to the tenets above. To begin with, we have the following rule of thumb for the amount of time you should be spending on each course.

Good Practice 1. Each week, spend three hours outside of class studying for every hour of class time.

So for a class which meets three times a week for an hour, you will want to spend nine hours studying each week. If your class meets twice a week for two hours, you will want to spend 12 hours studying each week. Etc. And here "studying" should be understood to include all of the following activities (and possibly more):

- Reading your notes.
- Reading the textbook (or lecture notes).
- Working through examples.
- Working on homework problems.
- Discussing mathematics with your peers.
- Discussing mathematics with your professor.
- Watching video lectures on line and taking notes.

Each of the above activities is probably worthy of being noted as a Good Practice. I'll highlight a few in particular.

Good Practice 2. Read the textbook (or lecture notes).

Here, "read" is to be interpreted as "actively read and understand." There is much more in your textbook than just homework exercises. In fact, the exposition (that is, the stuff between the theorems, definitions, examples, etc.) is usually quite good. A careful reading of this material can be enlightening.

In most mathematics classes, you will have homework assigned and collected regularly. In High School, it's common for students to develop the habit of starting an assignment the day before it is due. The demand in college is much higher, and as you progress in mathematics, the ideas become more complicated and take longer to master.

Good Practice 3. Begin your homework assignment the day it is assigned.

It may be that some of the material in your homework hasn't been covered in class yet, or maybe you haven't understood it well enough where you can fully solve each homework problem on the day homework is assigned. That's OK! Just by reading the assigned problems and understanding the statements of what is to be done can be incredibly useful. You'll be better prepared when the concepts are discussed in class and will even have a little more context for the material.

Good Practice 4. Turn your homework in.

Do this even if you don't finish it before the due date. Aside from getting whatever credit toward your grade your professor has decided each homework is worth, you'll get valuable feedback on the work you've done.

Good Practice 5. Review your homework when it is returned.

This practice will not only let you know what you have done correctly, but also will show you the mistakes you have made, giving you a chance to learn from them. Often, a professor (or whoever the homework grader is) will not grade every problem assigned, and so you may not have feedback on all of your work. Compare what you have received feedback on with those problems you have not. If you are lucky enough to have a professor who provides solutions to selected exercises after homework is collected, it's a good idea to read those solutions.

Good Practice 6. Review any exercise solutions provided by your professor.

Here "review" really means "read until you understand." If after a good reading effort you still don't understand, then talk with your peers and your professor to get an idea of how the solutions work.

Good Practice 7. Ask your peers and your professor for help when you don't understand something.

Abrupt ending to be resolved in a future draft.