Integration by the Wrong Parts

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1 Integration by Parts

In calculus, every rule for differentiation has a corresponding rule for anti-differentiation. To the Chain Rule corresponds the Substitution Rule, and to the Product Rule corresponds Integration by Parts, which is often written as follows:

$$\int u\, dv = uv - \int v\, dv.$$

When teaching integration by parts (IBP) in a college calculus class, it is often the role of the instructor to lead the students to make the "right" choices. Textbooks are filled with advice in this direction.

> In general, when deciding on a choice for u and dv, we usually try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x)dx$ can be readily integrated to give v. – Stewart, *Single Variable Calculus: Concepts and Contexts, 4e.*

> It helps if u' is simpler than u (or at least no more complicated than u). It helps if v is simpler than v' (or at least no more complicated than v'). – Hughes-Hallet, et al., *Calculus*, fifth edition.

Each of the texts quoted above (as do many, many others) computes $\int x \exp(x)\, dx$ to illustrate the "right" choice to make and the "wrong" choice to make. The seasoned veteran of anti-differentiation immediately sees that the "right" choice is to let $u = x$ and $dv = \exp(x)\, dx$, so that $du = dx$ and $v = \exp(x)$. Then we have

$$\begin{aligned} \int x \exp(x)\, dx &= x \exp(x) - \int \exp(x)\, dx \\ &= x \exp(x) - \exp(x) + C \end{aligned}$$

and we are done.

2 Using the Wrong Parts, a Lot

But, let's make the "wrong" choice and instead let $u_1 = \exp(x)$ and $dv_1 = x\, dx$, so that $du_1 = \exp(x)\, dx$ and $v_1 = \frac{1}{2}x^2$. Applying IBP yields the following:

$$\int x \exp(x)\, dx = \frac{1}{2}x^2 \exp(x) - \int \frac{1}{2}x^2 \exp(x)\, dx.$$

This is usually the point where instructors and textbook authors like to point out that the function we now need to find an anti-derivative for is more complicated than the one we started with, which indicates we are going the "wrong" way. But let's put our blinders on, keep going, and apply IBP again. Setting $u_2 = \exp(x)$ and $dv_2 = \frac{1}{2}x^2\ dx$, so that $du_2 = \exp(x)$ and $v_2 = \frac{1}{2\cdot 3}x^3$, gives the following:

$$\begin{aligned}\int x\exp(x)\ dx &= \frac{1}{2}x^2\exp(x) - \int \frac{1}{2}x^2\exp(x)\ dx \\ &= \frac{1}{2}x^2\exp(x) - \frac{1}{2\cdot 3}x^3\exp(x) + \int \frac{1}{2\cdot 3}x^3\exp(x)\ dx.\end{aligned}$$

Again, things are getting more complicated, and the "right" thing to do is start over and make "better" choices. But we will repeat the process again and again, next time choosing $u_3 = \exp(x)$ and $dv_3 = \frac{1}{2\cdot 3}x^3\ dx$. After n iterations, we arrive at the following:

$$\int x\exp(x)\ dx = \exp(x)\sum_{k=2}^{n+1}\frac{(-1)^k}{k!}x^k + \int \frac{(-1)^{n+1}}{(n+1)!}x^{n+1}\exp(x)\ dx.$$

We are building up a series, one application of IBP at a time. Taking the limit as $n\to\infty$ of this equation gives

$$\int x\exp(x)\ dx = \exp(x)\sum_{k=2}^{\infty}\frac{(-1)^k}{k!}x^k + \lim_{n\to\infty}\int \frac{(-1)^{n+1}}{(n+1)!}x^{n+1}\exp(x)\ dx.$$

Now we make use of our knowledge of the Taylor series for the exponential function, giving the following:

$$\int x\exp(x)\ dx = 1 - \exp(x) + x\exp(x) + \lim_{n\to\infty}\int \frac{(-1)^{n+1}}{(n+1)!}x^{n+1}\exp(x)\ dx.$$

The tempting thing to do now is to interchange the limit and anti-differentiation operations. To be certain we can do this, consider instead the following limit:

$$\lim_{n\to\infty}\int_0^x \frac{(-1)^{n+1}}{(n+1)!}t^{n+1}\exp(t)\ dt.$$

The sequence of functions $\left|\frac{(-1)^{n+1}}{(n+1)!}t^{n+1}\exp(t)\right|$ converges uniformly to 0 on $[0,x]$ (or $[x,0]$ if $x<0$). Hence we can interchange the limit and integration operators, giving

$$\begin{aligned}\lim_{n\to\infty}\int_0^x \frac{(-1)^{n+1}}{(n+1)!}t^{n+1}\exp(t)\ dt &= \int_0^x \lim_{n\to\infty}\frac{(-1)^{n+1}}{(n+1)!}t^{n+1}\exp(t)\ dt \\ &= \int_0^x 0\ dt \\ &= 0.\end{aligned}$$

Hence we have the following:

$$\lim_{n\to\infty}\int \frac{(-1)^{n+1}}{(n+1)!}x^{n+1}\exp(x)\ dx = C.$$

Returning to our main computation, we get

$$\begin{aligned}\int x\exp(x)\ dx &= 1-\exp(x)+x\exp(x)+\lim_{n\to\infty}\int \frac{(-1)^{n+1}}{(n+1)!}x^{n+1}\exp(x)\ dx\\ &= 1-\exp(x)+x\exp(x)+C\\ &= x\exp(x)-\exp(x)+C.\end{aligned}$$

Hence, we have successfully applied Integration by the Wrong Parts (IBWP).

3 The Taylor Series for $\exp(x)$

When we computed $\int x\exp(x)\ dx$ using IBWP, we made use of the fact that the Taylor series for the exponential function was already known. We can actually use IBWP to derive this series. Starting with $\int \exp(x)\ dx$ and applying IBWP with $u_1=\exp(x)$ and $dv_1=dx$, we arrive at

$$\int \exp(x)\ dx = -\exp(x)\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}x^n + C.$$

So for some value of C, we have

$$\exp(x) = -\exp(x)\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}x^n + C.$$

Letting $x=0$, we see that $C=1$. Dividing by $\exp(x)$ gives us

$$1 = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}x^n + \exp(-x)$$

in which we can solve for $\exp(-x)$ to get

$$\exp(-x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n.$$

Now replacing x with $-x$ gives us

$$\exp(x) = \sum_{n=0}^{\infty}\frac{1}{n!}x^n.$$

4 Further Examples

Applying IBWP to each of $\int \sin(x)\ dx$ and $\int \cos(x)\ dx$ yields the Taylor series for sine and cosine. In fact, if we let $\sigma_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$ and $\sigma_2(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$, then IBWP produces the system of equations below.

$$\begin{aligned}0 &= \sigma_1(x)\cos(x)-\sigma_2(x)\sin(x)\\ 1 &= \sigma_1(x)\sin(x)+\sigma_2(x)\cos(x)\end{aligned}$$

These can then be solved to give us $\sigma_1(x)=\sin(x)$ and $\sigma_2(x)=\cos(x)$.

If you are looking for more fun examples, try applying IBWP to compute $\int x\sin(x)\ dx$ and $\int x\cos(x)\ dx$. Once you have those down, you can also try the corresponding anti-derivatives involving the hyperbolic sine and hyperbolic cosine functions.

5 Conclusion

While two wrongs do not make a right, sometimes an infinite sequence of wrongs does.

Abstract

In this article, we disregard common sense and good advice and compute anti-derivatives for certain functions by stubbornly applying integration by parts infinitely many times.

References

[1] D. Hughes-Hallet, A. M. Gleason, W. G.McCallum, et al., *Calculus*. John Wiley & Sons, Inc., Hoboken, NJ, 2009.

[2] J. Stewart, *Single Variable Calculus, Concepts and Contexts*. Brooks/Cole, Cengage Learning, Belmont, CA, 2010.