Convex Neural Codes

Bill Kronholm

Whittier College

UCLA Topology Seminar
March 16, 2016

Hippocampus

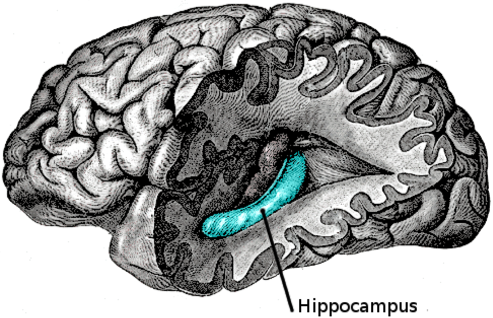


Image credit: Henry Vandyke Carter (1919)

Place Cells

- Place cells are a particular type of neuron found in the hippocampus.
- Individual place cells are active when the creature is in a particular location in the environment called the **place field** or **receptive field** for the neuron.
- Reasonable hypothesis: The receptive field for a particular place cell is a convex region in some space.

Place Cells

- Place cells are a particular type of neuron found in the hippocampus.
- Individual place cells are active when the creature is in a particular location in the environment called the **place field** or **receptive field** for the neuron.
- Reasonable hypothesis: The receptive field for a particular place cell is a convex region in some space.

Place Cells

- Place cells are a particular type of neuron found in the hippocampus.
- Individual place cells are active when the creature is in a particular location in the environment called the **place field** or **receptive field** for the neuron.
- Reasonable hypothesis: The receptive field for a particular place cell is a convex region in some space.

Example

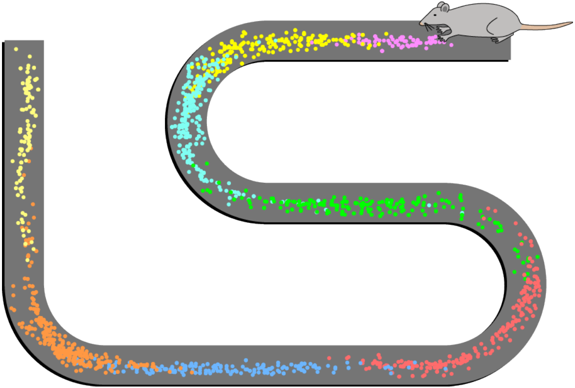


Image credit: “Place Cell Spiking Activity Example” by Stuartlayton, CC BY-SA 3.0

Neural Codes

- A **neural code** is the collection of firing patterns of a group of neurons.
- Idea: abstract the neurons and firing patterns to an appropriate mathematical model and study the model.

Neural Codes

- A **neural code** is the collection of firing patterns of a group of neurons.
- Idea: abstract the neurons and firing patterns to an appropriate mathematical model and study the model.

Definitions

- $[n] = \{1, 2, \ldots, n\}$
- An element $i \in [n]$ is a **neuron**.
- $2^{[n]} = \mathcal{P}([n])$
- A **code** $\mathcal{C}$ is a subset $\mathcal{C} \subset 2^{[n]}$.
- An element $\sigma \in \mathcal{C}$ is a **codeword** of $\mathcal{C}$. Each codeword represents a collection of neurons which co-fire.

Definitions

- $[n] = \{1, 2, \ldots, n\}$
- An element $i \in [n]$ is a **neuron**.
- $2^{[n]} = \mathcal{P}([n])$
- A **code** $\mathcal{C}$ is a subset $\mathcal{C} \subset 2^{[n]}$.
- An element $\sigma \in \mathcal{C}$ is a **codeword** of $\mathcal{C}$. Each codeword represents a collection of neurons which co-fire.

Definitions

- $[n] = \{1, 2, \ldots, n\}$
- An element $i \in [n]$ is a **neuron**.
- $2^{[n]} = \mathcal{P}([n])$
- A **code** $\mathcal{C}$ is a subset $\mathcal{C} \subset 2^{[n]}$.
- An element $\sigma \in \mathcal{C}$ is a **codeword** of $\mathcal{C}$. Each codeword represents a collection of neurons which co-fire.

Definitions

- $[n] = \{1, 2, \ldots, n\}$
- An element $i \in [n]$ is a **neuron**.
- $2^{[n]} = \mathcal{P}([n])$
- A **code** $\mathcal{C}$ is a subset $\mathcal{C} \subset 2^{[n]}$.
- An element $\sigma \in \mathcal{C}$ is a **codeword** of $\mathcal{C}$. Each codeword represents a collection of neurons which co-fire.

Definitions

- $[n] = \{1, 2, \ldots, n\}$
- An element $i \in [n]$ is a **neuron**.
- $2^{[n]} = \mathcal{P}([n])$
- A **code** $\mathcal{C}$ is a subset $\mathcal{C} \subset 2^{[n]}$.
- An element $\sigma \in \mathcal{C}$ is a **codeword** of $\mathcal{C}$. Each codeword represents a collection of neurons which co-fire.

Definitions

- A **cover** is a collection $\mathcal{U} = \{U(i)\}_{i \in [n]}$ of subsets U_i of $\mathbb{R}^d$.
- The **code** of a cover $\mathcal{U} = \{U(i)\}_{i \in [n]}$ is the set $\text{code}(\mathcal{U}) \subset 2^{[n]}$ given by

$$\sigma \in \text{code}(\mathcal{U}) \Leftrightarrow \left(\bigcap_{i \in \sigma} U(i) \right) \setminus \left(\bigcup_{j \in [n] \setminus \sigma} U(j) \right) \neq \varnothing.$$

- Such a non-empty region is called the **atom** for σ.

Definitions

- A **cover** is a collection $\mathcal{U} = \{U(i)\}_{i\in[n]}$ of subsets U_i of $\mathbb{R}^d$.
- The **code** of a cover $\mathcal{U} = \{U(i)\}_{i\in[n]}$ is the set $\text{code}(\mathcal{U}) \subset 2^{[n]}$ given by

$$\sigma \in \text{code}(\mathcal{U}) \Leftrightarrow \left(\bigcap_{i\in\sigma} U(i)\right) \setminus \left(\bigcup_{j\in[n]\setminus\sigma} U(j)\right) \neq \varnothing.$$

- Such a non-empty region is called the **atom** for σ.

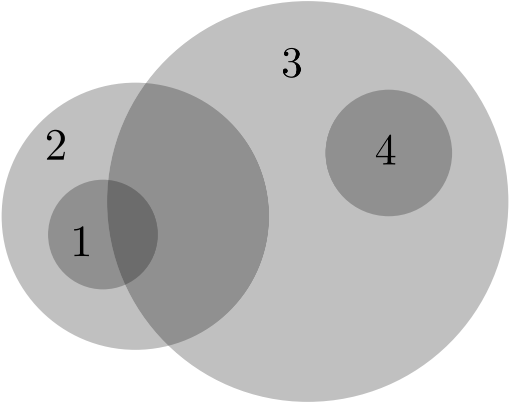
Definitions

- A **cover** is a collection $\mathcal{U} = \{U(i)\}_{i \in [n]}$ of subsets U_i of $\mathbb{R}^d$.
- The **code** of a cover $\mathcal{U} = \{U(i)\}_{i \in [n]}$ is the set $\text{code}(\mathcal{U}) \subset 2^{[n]}$ given by

$$\sigma \in \text{code}(\mathcal{U}) \Leftrightarrow \left(\bigcap_{i \in \sigma} U(i) \right) \setminus \left(\bigcup_{j \in [n] \setminus \sigma} U(j) \right) \neq \varnothing.$$

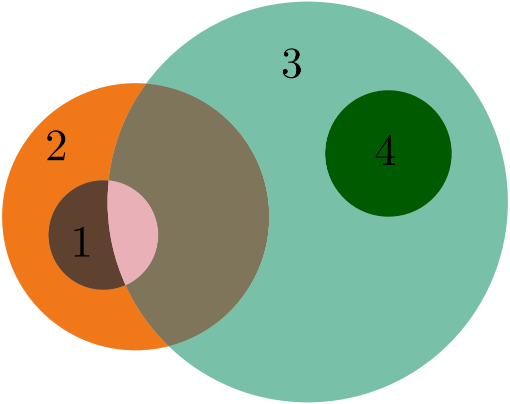
- Such a non-empty region is called the **atom** for σ.

Example



$$\mathcal{C} = \{2, 3, 12, 23, 34, 123\}$$

Example



$$\mathcal{C} = \{2, 3, 12, 23, 34, 123\}$$

Convex Codes

A code $\mathcal{C}$ is **open convex** if it can be realized as the code of a cover of open convex sets in $\mathbb{R}^d$ for some d.

A code $\mathcal{C}$ is **closed convex** if it can be realized as the code of a cover of closed convex sets in $\mathbb{R}^d$ for some d.

Main question: Without constructing a cover, how can we determine whether or not a given code is open/closed convex?

Convex Codes

A code $\mathcal{C}$ is **open convex** if it can be realized as the code of a cover of open convex sets in $\mathbb{R}^d$ for some d.
A code $\mathcal{C}$ is **closed convex** if it can be realized as the code of a cover of closed convex sets in $\mathbb{R}^d$ for some d.

Main question: Without constructing a cover, how can we determine whether or not a given code is open/closed convex?

Convex Codes

A code $\mathcal{C}$ is **open convex** if it can be realized as the code of a cover of open convex sets in $\mathbb{R}^d$ for some d.
A code $\mathcal{C}$ is **closed convex** if it can be realized as the code of a cover of closed convex sets in $\mathbb{R}^d$ for some d.

Main question: Without constructing a cover, how can we determine whether or not a given code is open/closed convex?

Simplicial Complex

A set $\mathcal{C} \subset 2^{[n]}$ is a **simplicial complex** if it is closed under taking subsets. That is, if $\sigma \in \mathcal{C}$ and $\tau \subset \sigma$, then $\tau \in \mathcal{C}$.

For example, $\mathcal{C} = \{2, 3, 13, 23, 123\}$ is not a simplicial complex. $(13 \in \mathcal{C}$ but $1 \notin \mathcal{C})$

Simplicial Complex

A set $\mathcal{C} \subset 2^{[n]}$ is a **simplicial complex** if it is closed under taking subsets. That is, if $\sigma \in \mathcal{C}$ and $\tau \subset \sigma$, then $\tau \in \mathcal{C}$.

For example, $\mathcal{C} = \{2, 3, 13, 23, 123\}$ is not a simplicial complex. ($13 \in \mathcal{C}$ but $1 \notin \mathcal{C}$)

Simplicial Complex

A set $\mathcal{C} \subset 2^{[n]}$ is a **simplicial complex** if it is closed under taking subsets. That is, if $\sigma \in \mathcal{C}$ and $\tau \subset \sigma$, then $\tau \in \mathcal{C}$.

For example, $\mathcal{C} = \{2, 3, 13, 23, 123\}$ is not a simplicial complex.
$(13 \in \mathcal{C}$ but $1 \notin \mathcal{C})$

Simplicial Complex

A set $\mathcal{C} \subset 2^{[n]}$ is a **simplicial complex** if it is closed under taking subsets. That is, if $\sigma \in \mathcal{C}$ and $\tau \subset \sigma$, then $\tau \in \mathcal{C}$.

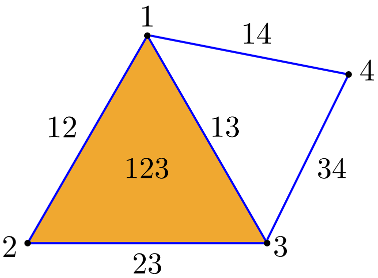
For example, $\mathcal{C} = \{2, 3, 13, 23, 123\}$ is not a simplicial complex. ($13 \in \mathcal{C}$ but $1 \notin \mathcal{C}$)

Simplicial Complex

A simplicial complex can be realized as a topological space.

Simplicial Complex

A simplicial complex can be realized as a topological space.



$$\mathcal{C} = \{1, 2, 3, 12, 13, 14, 23, 34, 123\}$$

Simplicial Violators

Given a code $\mathcal{C}$ on $[n]$, let $\Delta(\mathcal{C})$ denote the smallest simplicial complex on $[n]$ containing $\mathcal{C}$.

Any $\sigma \in \Delta(\mathcal{C}) \setminus \mathcal{C}$ is called a **simplicial violator** of $\mathcal{C}$.

If σ is a simplicial violator of $\mathcal{C}$, then the **localization** of $\mathcal{C}$ at σ is the code $\mathcal{C}|_\sigma$ on the set of neurons $[n] \setminus \sigma$ where $\tau \in \mathcal{C}|_\sigma$ if and only if $\tau \cup \sigma \in \mathcal{C}$.

Simplicial Violators

Given a code $\mathcal{C}$ on $[n]$, let $\Delta(\mathcal{C})$ denote the smallest simplicial complex on $[n]$ containing $\mathcal{C}$.
Any $\sigma \in \Delta(\mathcal{C}) \setminus \mathcal{C}$ is called a **simplicial violator** of $\mathcal{C}$.
If σ is a simplicial violator of $\mathcal{C}$, then the **localization** of $\mathcal{C}$ at σ is the code $\mathcal{C}|_\sigma$ on the set of neurons $[n] \setminus \sigma$ where $\tau \in \mathcal{C}|_\sigma$ if and only if $\tau \cup \sigma \in \mathcal{C}$.

Simplicial Violators

Given a code $\mathcal{C}$ on $[n]$, let $\Delta(\mathcal{C})$ denote the smallest simplicial complex on $[n]$ containing $\mathcal{C}$.
Any $\sigma \in \Delta(\mathcal{C}) \setminus \mathcal{C}$ is called a **simplicial violator** of $\mathcal{C}$.
If σ is a simplicial violator of $\mathcal{C}$, then the **localization** of $\mathcal{C}$ at σ is the code $\mathcal{C}|_\sigma$ on the set of neurons $[n] \setminus \sigma$ where $\tau \in \mathcal{C}|_\sigma$ if and only if $\tau \cup \sigma \in \mathcal{C}$.

Example

Let $\mathcal{C} = \{2, 3, 13, 23, 123\}$. Then $\sigma = 1$ is a simplicial violator for $\mathcal{C}$ and $\mathcal{C}|_1 = \{3, 23\}$. Also, $\tau = 12$ is a simplicial violator, and $\mathcal{C}|_{12} = \{3\}$.

Known Local Obstruction

Theorem (C. Giusti, V. Itskov)

Let σ be a simplicial violator of $\mathcal{C}$. If $\Delta(\mathcal{C}|_\sigma)$ is not a contractible simplicial complex, then $\mathcal{C}$ is not convex.

Assume $\Delta(\mathcal{C}|_\sigma) \not\simeq *$ but $\mathcal{C}$ is open (or closed) convex.
Let $\mathcal{U} = \{U_i\}_{i\in[n]}$ be an open (closed) convex cover so that code$(\mathcal{U}) = \mathcal{C}$.

- Denote $U_\sigma = \cap_{i\in\sigma} U_i$, and $V_j = U_j \cap U_\sigma$.
- Each V_j is open (closed) and convex.
- $\tau \in \Delta(\mathcal{C}|_\sigma)$ if and only if $V_\tau = \cap_{j\in\tau} V_j$ is non-empty.
- $U_\sigma = \cup_{j\in[n]\setminus\sigma} V_j \quad$ (b/c $\sigma \notin \mathcal{C}$)

Known Local Obstruction

Theorem (C. Giusti, V. Itskov)

Let σ be a simplicial violator of $\mathcal{C}$. If $\Delta(\mathcal{C}|_\sigma)$ is not a contractible simplicial complex, then $\mathcal{C}$ is not convex.

Assume $\Delta(\mathcal{C}|_\sigma) \not\simeq *$ but $\mathcal{C}$ is open (or closed) convex.

Let $\mathcal{U} = \{U_i\}_{i\in[n]}$ be an open (closed) convex cover so that $\text{code}(\mathcal{U}) = \mathcal{C}$.

- Denote $U_\sigma = \cap_{i\in\sigma} U_i$, and $V_j = U_j \cap U_\sigma$.
- Each V_j is open (closed) and convex.
- $\tau \in \Delta(\mathcal{C}|_\sigma)$ if and only if $V_\tau = \cap_{j\in\tau} V_j$ is non-empty.
- $U_\sigma = \cup_{j\in[n]\setminus\sigma} V_j$ (b/c $\sigma \notin \mathcal{C}$)

Known Local Obstruction

Theorem (C. Giusti, V. Itskov)

Let σ be a simplicial violator of $\mathcal{C}$. If $\Delta(\mathcal{C}|_\sigma)$ is not a contractible simplicial complex, then $\mathcal{C}$ is not convex.

Assume $\Delta(\mathcal{C}|_\sigma) \not\simeq *$ but $\mathcal{C}$ is open (or closed) convex.
Let $\mathcal{U} = \{U_i\}_{i\in[n]}$ be an open (closed) convex cover so that $\text{code}(\mathcal{U}) = \mathcal{C}$.

- Denote $U_\sigma = \cap_{i\in\sigma} U_i$, and $V_j = U_j \cap U_\sigma$.
- Each V_j is open (closed) and convex.
- $\tau \in \Delta(\mathcal{C}|_\sigma)$ if and only if $V_\tau = \cap_{j\in\tau} V_j$ is non-empty.
- $U_\sigma = \cup_{j\in[n]\setminus\sigma} V_j$ (b/c $\sigma \notin \mathcal{C}$)

Known Local Obstruction

Theorem (C. Giusti, V. Itskov)

Let σ be a simplicial violator of $\mathcal{C}$. If $\Delta(\mathcal{C}|_\sigma)$ is not a contractible simplicial complex, then $\mathcal{C}$ is not convex.

Assume $\Delta(\mathcal{C}|_\sigma) \not\simeq *$ but $\mathcal{C}$ is open (or closed) convex.
Let $\mathcal{U} = \{U_i\}_{i\in[n]}$ be an open (closed) convex cover so that $\text{code}(\mathcal{U}) = \mathcal{C}$.

- Denote $U_\sigma = \cap_{i\in\sigma} U_i$, and $V_j = U_j \cap U_\sigma$.
- Each V_j is open (closed) and convex.
- $\tau \in \Delta(\mathcal{C}|_\sigma)$ if and only if $V_\tau = \cap_{j\in\tau} V_j$ is non-empty.
- $U_\sigma = \cup_{j\in[n]\setminus\sigma} V_j$ (b/c $\sigma \notin \mathcal{C}$)

Known Local Obstruction

Theorem (C. Giusti, V. Itskov)

Let σ be a simplicial violator of $\mathcal{C}$. If $\Delta(\mathcal{C}|_\sigma)$ is not a contractible simplicial complex, then $\mathcal{C}$ is not convex.

Assume $\Delta(\mathcal{C}|_\sigma) \not\simeq *$ but $\mathcal{C}$ is open (or closed) convex.
Let $\mathcal{U} = \{U_i\}_{i\in[n]}$ be an open (closed) convex cover so that $\text{code}(\mathcal{U}) = \mathcal{C}$.

- Denote $U_\sigma = \cap_{i\in\sigma} U_i$, and $V_j = U_j \cap U_\sigma$.
- Each V_j is open (closed) and convex.
- $\tau \in \Delta(\mathcal{C}|_\sigma)$ if and only if $V_\tau = \cap_{j\in\tau} V_j$ is non-empty.
- $U_\sigma = \cup_{j\in[n]\setminus\sigma} V_j$ (b/c $\sigma \notin \mathcal{C}$)

Known Local Obstruction

Theorem (C. Giusti, V. Itskov)

Let σ be a simplicial violator of $\mathcal{C}$. If $\Delta(\mathcal{C}|_\sigma)$ is not a contractible simplicial complex, then $\mathcal{C}$ is not convex.

Assume $\Delta(\mathcal{C}|_\sigma) \not\simeq *$ but $\mathcal{C}$ is open (or closed) convex.
Let $\mathcal{U} = \{U_i\}_{i\in[n]}$ be an open (closed) convex cover so that $\text{code}(\mathcal{U}) = \mathcal{C}$.

- Denote $U_\sigma = \cap_{i\in\sigma} U_i$, and $V_j = U_j \cap U_\sigma$.
- Each V_j is open (closed) and convex.
- $\tau \in \Delta(\mathcal{C}|_\sigma)$ if and only if $V_\tau = \cap_{j\in\tau} V_j$ is non-empty.
- $U_\sigma = \cup_{j\in[n]\setminus\sigma} V_j$ (b/c $\sigma \notin \mathcal{C}$)

Known Local Obstruction

Theorem (C. Giusti, V. Itskov)

Let σ be a simplicial violator of $\mathcal{C}$. If $\Delta(\mathcal{C}|_\sigma)$ is not a contractible simplicial complex, then $\mathcal{C}$ is not convex.

Assume $\Delta(\mathcal{C}|_\sigma) \not\simeq *$ but $\mathcal{C}$ is open (or closed) convex.
Let $\mathcal{U} = \{U_i\}_{i\in[n]}$ be an open (closed) convex cover so that $\text{code}(\mathcal{U}) = \mathcal{C}$.

- Denote $U_\sigma = \cap_{i\in\sigma} U_i$, and $V_j = U_j \cap U_\sigma$.
- Each V_j is open (closed) and convex.
- $\tau \in \Delta(\mathcal{C}|_\sigma)$ if and only if $V_\tau = \cap_{j\in\tau} V_j$ is non-empty.
- $U_\sigma = \cup_{j\in[n]\setminus\sigma} V_j \quad$ (b/c $\sigma \notin \mathcal{C}$)

Known Local Obstruction

- $\Delta(\mathcal{C}|_\sigma)$ is the nerve of the cover of U_σ by open (closed) convex sets V_j.
- Therefore, by the nerve lemma, $U_\sigma \simeq \Delta(\mathcal{C}|_\sigma)$.
- $U_\sigma \simeq *$, but $\Delta(\mathcal{C}|_\sigma) \not\simeq *$
- Hence $\mathcal{C}$ is not open (closed) convex.

Known Local Obstruction

- $\Delta(\mathcal{C}|_\sigma)$ is the nerve of the cover of U_σ by open (closed) convex sets V_j.
- Therefore, by the nerve lemma, $U_\sigma \simeq \Delta(\mathcal{C}|_\sigma)$.
- $U_\sigma \simeq *$, but $\Delta(\mathcal{C}|_\sigma) \not\simeq *$
- Hence $\mathcal{C}$ is not open (closed) convex.

Known Local Obstruction

- $\Delta(\mathcal{C}|_\sigma)$ is the nerve of the cover of U_σ by open (closed) convex sets V_j.
- Therefore, by the nerve lemma, $U_\sigma \simeq \Delta(\mathcal{C}|_\sigma)$.
- $U_\sigma \simeq *$, but $\Delta(\mathcal{C}|_\sigma) \not\simeq *$
- Hence $\mathcal{C}$ is not open (closed) convex.

Known Local Obstruction

- $\Delta(\mathcal{C}|_\sigma)$ is the nerve of the cover of U_σ by open (closed) convex sets V_j.
- Therefore, by the nerve lemma, $U_\sigma \simeq \Delta(\mathcal{C}|_\sigma)$.
- $U_\sigma \simeq *$, but $\Delta(\mathcal{C}|_\sigma) \not\simeq *$
- Hence $\mathcal{C}$ is not open (closed) convex.

Example

- $\mathcal{C} = \{1, 2, 12, 13, 23\}$
- $\sigma = 3$ is a simplicial violator.
- $\mathcal{C}|_\sigma = \{1, 2\} = \Delta(\mathcal{C}|_\sigma) \not\simeq *$
- So $\mathcal{C}$ is **not** a convex code.

Example

- $\mathcal{C} = \{1, 2, 12, 13, 23\}$
- $\sigma = 3$ is a simplicial violator.
- $\mathcal{C}|_\sigma = \{1, 2\} = \Delta(\mathcal{C}|_\sigma) \not\simeq *$
- So $\mathcal{C}$ is **not** a convex code.

Example

- $\mathcal{C} = \{1, 2, 12, 13, 23\}$
- $\sigma = 3$ is a simplicial violator.
- $\mathcal{C}|_\sigma = \{1, 2\} = \Delta(\mathcal{C}|_\sigma) \not\simeq *$
- So $\mathcal{C}$ is **not** a convex code.

Example

- $\mathcal{C} = \{1, 2, 12, 13, 23\}$
- $\sigma = 3$ is a simplicial violator.
- $\mathcal{C}|_\sigma = \{1, 2\} = \Delta(\mathcal{C}|_\sigma) \not\simeq *$
- So $\mathcal{C}$ is **not** a convex code.

Question

If $\Delta(\mathcal{C}|_\sigma)$ is contractible for every simplicial violator σ, is $\mathcal{C}$ open (closed) convex?

The answer turns out to be no for open convex:The code $\mathcal{C} = \{1, 3, 12, 13, 15, 34, 35, 125, 135, 345, 1234\}$ is not open convex, but $\mathcal{C}|_\sigma \simeq *$ for all simplicial violators σ.
[Lienkaemper] (Texas A&M REU, Summer 2015)

Question

If $\Delta(\mathcal{C}|_\sigma)$ is contractible for every simplicial violator σ, is $\mathcal{C}$ open (closed) convex?

The answer turns out to be no for open convex: The code $\mathcal{C} = \{1, 3, 12, 13, 15, 34, 35, 125, 135, 345, 1234\}$ is not open convex, but $\mathcal{C}|_\sigma \simeq *$ for all simplicial violators σ.
[Lienkaemper] (Texas A&M REU, Summer 2015)

Question

If $\Delta(\mathcal{C}|_\sigma)$ is contractible for every simplicial violator σ, is $\mathcal{C}$ open (closed) convex?

The answer turns out to be no for open convex:The code $\mathcal{C} = \{1, 3, 12, 13, 15, 34, 35, 125, 135, 345, 1234\}$ is not open convex, but $\mathcal{C}|_\sigma \simeq *$ for all simplicial violators σ.

[Lienkaemper] (Texas A&M REU, Summer 2015)

Question

If $\Delta(\mathcal{C}|_\sigma)$ is contractible for every simplicial violator σ, is $\mathcal{C}$ open (closed) convex?

The answer turns out to be no for open convex:The code $\mathcal{C} = \{1, 3, 12, 13, 15, 34, 35, 125, 135, 345, 1234\}$ is not open convex, but $\mathcal{C}|_\sigma \simeq *$ for all simplicial violators σ.
[Lienkaemper] (Texas A&M REU, Summer 2015)

Contractibility

- A (finite) simplicial complex K is contractible if and only if all homology groups are trivial and the fundamental group is trivial.
- Homology is easy to compute.
- A presentation for the fundamental group is easy to write down from the data of the simplicial complex.
- However, it is undecidable whether the presentation is the trivial group or not. This is the word problem for groups.

Contractibility

- A (finite) simplicial complex K is contractible if and only if all homology groups are trivial and the fundamental group is trivial.
- Homology is easy to compute.
- A presentation for the fundamental group is easy to write down from the data of the simplicial complex.
- However, it is undecidable whether the presentation is the trivial group or not. This is the word problem for groups.

Contractibility

- A (finite) simplicial complex K is contractible if and only if all homology groups are trivial and the fundamental group is trivial.
- Homology is easy to compute.
- A presentation for the fundamental group is easy to write down from the data of the simplicial complex.
- However, it is undecidable whether the presentation is the trivial group or not. This is the word problem for groups.

Contractibility

- A (finite) simplicial complex K is contractible if and only if all homology groups are trivial and the fundamental group is trivial.
- Homology is easy to compute.
- A presentation for the fundamental group is easy to write down from the data of the simplicial complex.
- However, it is undecidable whether the presentation is the trivial group or not. This is the word problem for groups.

Contractibility

- A (finite) simplicial complex K is contractible if and only if all homology groups are trivial and the fundamental group is trivial.
- Homology is easy to compute.
- A presentation for the fundamental group is easy to write down from the data of the simplicial complex.
- However, it is undecidable whether the presentation is the trivial group or not. This is the word problem for groups.

Intersection Completion

Given a code $\mathcal{C}$, let $\widehat{\mathcal{C}} = \{\sigma \mid \sigma = \bigcap_{\tau \in X} \tau \text{ for some } X \subset 2^{\mathcal{C}}\}$ be the **intersection completion of** $\mathcal{C}$.

If $\mathcal{C} = \widehat{\mathcal{C}}$, then $\mathcal{C}$ is **intersection complete**.

Theorem (GIK)

Intersection complete codes are convex.

Corollary

Simplicial complexes are convex codes.

Intersection Completion

Given a code $\mathcal{C}$, let $\widehat{\mathcal{C}} = \{\sigma \mid \sigma = \bigcap_{\tau \in X} \tau \text{ for some } X \subset 2^{\mathcal{C}}\}$ be the **intersection completion of** $\mathcal{C}$.
If $\mathcal{C} = \widehat{\mathcal{C}}$, then $\mathcal{C}$ is **intersection complete**.

Theorem (GIK)

Intersection complete codes are convex.

Corollary

Simplicial complexes are convex codes.

Intersection Completion

Given a code $\mathcal{C}$, let $\widehat{\mathcal{C}} = \{\sigma \mid \sigma = \bigcap_{\tau \in X} \tau \text{ for some } X \subset 2^{\mathcal{C}}\}$ be the **intersection completion of** $\mathcal{C}$.
If $\mathcal{C} = \widehat{\mathcal{C}}$, then $\mathcal{C}$ is **intersection complete**.

Theorem (GIK)

Intersection complete codes are convex.

Corollary

Simplicial complexes are convex codes.

A Cover for Intersection Complete Codes

Let $\mathcal{C}$ be a code on $[n]$.
For each $\sigma \in \mathcal{C}$ let e_σ be a unit vector in $\mathbb{R}^{|\mathcal{C}|}$ so that $\{e_\sigma\}$ is a basis for $\mathbb{R}^{|\mathcal{C}|}$.
For each $i \in [n] = \{1, \ldots, n\}$, let $v(i)$ be the following set of points:

$$v(i) = \{e_\sigma \mid \sigma \in \mathcal{C},\ i \in \sigma\}.$$

Define $P(i) = \text{conv}(v(i))$ to be the convex hull of $v(i)$.
The collection $\text{cov}(\mathcal{C}) = \{P(i)\}_{i \in [n]}$ is the **potential cover** for $\mathcal{C}$.

A Cover for Intersection Complete Codes

Let $\mathcal{C}$ be a code on $[n]$.
For each $\sigma \in \mathcal{C}$ let e_σ be a unit vector in $\mathbb{R}^{|\mathcal{C}|}$ so that $\{e_\sigma\}$ is a basis for $\mathbb{R}^{|\mathcal{C}|}$.
For each $i \in [n] = \{1, \ldots, n\}$, let $v(i)$ be the following set of points:

$$v(i) = \{e_\sigma \mid \sigma \in \mathcal{C},\ i \in \sigma\}.$$

Define $P(i) = \operatorname{conv}(v(i))$ to be the convex hull of $v(i)$.
The collection $\operatorname{cov}(\mathcal{C}) = \{P(i)\}_{i \in [n]}$ is the **potential cover** for $\mathcal{C}$.

A Cover for Intersection Complete Codes

Let $\mathcal{C}$ be a code on $[n]$.
For each $\sigma \in \mathcal{C}$ let e_σ be a unit vector in $\mathbb{R}^{|\mathcal{C}|}$ so that $\{e_\sigma\}$ is a basis for $\mathbb{R}^{|\mathcal{C}|}$.
For each $i \in [n] = \{1, \ldots, n\}$, let $v(i)$ be the following set of points:

$$v(i) = \{e_\sigma \mid \sigma \in \mathcal{C},\ i \in \sigma\}.$$

Define $P(i) = \text{conv}(v(i))$ to be the convex hull of $v(i)$.
The collection $\text{cov}(\mathcal{C}) = \{P(i)\}_{i \in [n]}$ is the **potential cover** for $\mathcal{C}$.

A Cover for Intersection Complete Codes

Let $\mathcal{C}$ be a code on $[n]$.
For each $\sigma \in \mathcal{C}$ let e_σ be a unit vector in $\mathbb{R}^{|\mathcal{C}|}$ so that $\{e_\sigma\}$ is a basis for $\mathbb{R}^{|\mathcal{C}|}$.
For each $i \in [n] = \{1, \ldots, n\}$, let $v(i)$ be the following set of points:

$$v(i) = \{e_\sigma \mid \sigma \in \mathcal{C},\ i \in \sigma\}.$$

Define $P(i) = \text{conv}(v(i))$ to be the convex hull of $v(i)$.
The collection $\text{cov}(\mathcal{C}) = \{P(i)\}_{i \in [n]}$ is the **potential cover** for $\mathcal{C}$.

A Cover for Intersection Complete Codes

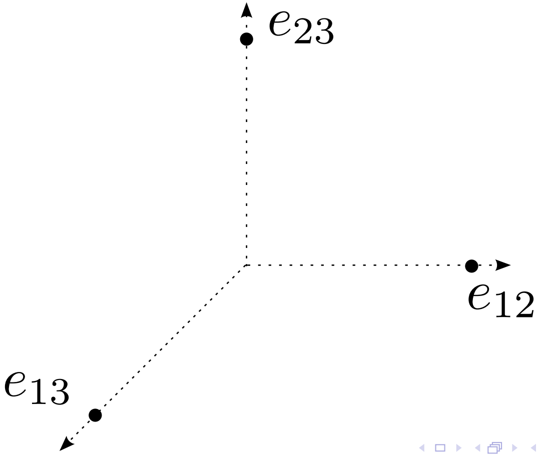
Let $\mathcal{C}$ be a code on $[n]$.
For each $\sigma \in \mathcal{C}$ let e_σ be a unit vector in $\mathbb{R}^{|\mathcal{C}|}$ so that $\{e_\sigma\}$ is a basis for $\mathbb{R}^{|\mathcal{C}|}$.
For each $i \in [n] = \{1, \ldots, n\}$, let $v(i)$ be the following set of points:

$$v(i) = \{e_\sigma \mid \sigma \in \mathcal{C},\ i \in \sigma\}.$$

Define $P(i) = \text{conv}(v(i))$ to be the convex hull of $v(i)$.
The collection $\text{cov}(\mathcal{C}) = \{P(i)\}_{i \in [n]}$ is the **potential cover** for $\mathcal{C}$.

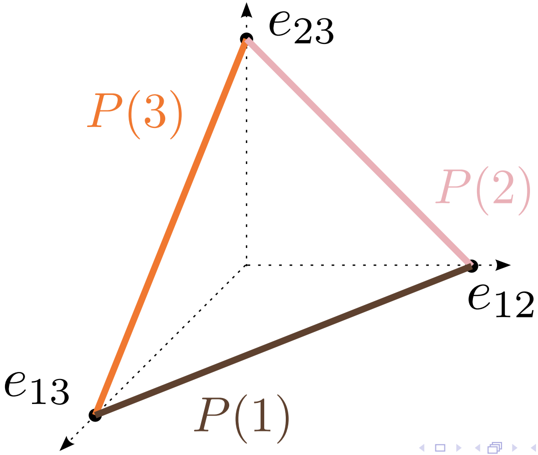
Example

$$\mathcal{C} = \{12, 13, 23, 123\}$$



Example

$$\mathcal{C} = \{12, 13, 23, 123\}$$



A Cover for Intersection Complete Codes

Theorem (GIK)

$\text{code}(\text{cov}(\mathcal{C})) = \widehat{\mathcal{C}}$

- $\sigma \in \text{code}(\text{cov}(\mathcal{C})) \Rightarrow (\cap_{i \in \sigma} P(i)) \setminus \cup_{j \notin \sigma} P(j) \neq \varnothing$
- $j \in \bigcap_{\sigma \subset \tau} (\tau \setminus \sigma) \Rightarrow \bigcap_{i \in \sigma} P(i) \subset P(j)$
- Thus $\sigma = \bigcap_{\tau \in \mathcal{C}, \sigma \subset \tau} \tau$, so $\sigma \in \widehat{\mathcal{C}}$

A Cover for Intersection Complete Codes

Theorem (GIK)

$\text{code}(\text{cov}(\mathcal{C})) = \widehat{\mathcal{C}}$

- $\sigma \in \text{code}(\text{cov}(\mathcal{C})) \Rightarrow (\cap_{i\in\sigma} P(i)) \setminus \cup_{j\notin\sigma} P(j) \neq \varnothing$
- $j \in \bigcap_{\sigma\subset\tau}(\tau \setminus \sigma) \Rightarrow \bigcap_{i\in\sigma} P(i) \subset P(j)$
- Thus $\sigma = \bigcap_{\tau\in\mathcal{C},\sigma\subset\tau} \tau$, so $\sigma \in \widehat{\mathcal{C}}$

A Cover for Intersection Complete Codes

Theorem (GIK)

$\text{code}(\text{cov}(\mathcal{C})) = \widehat{\mathcal{C}}$

- $\sigma \in \text{code}(\text{cov}(\mathcal{C})) \Rightarrow (\cap_{i\in\sigma} P(i)) \setminus \cup_{j\notin\sigma} P(j) \neq \varnothing$
- $j \in \bigcap_{\sigma\subset\tau}(\tau \setminus \sigma) \Rightarrow \bigcap_{i\in\sigma} P(i) \subset P(j)$
- Thus $\sigma = \bigcap_{\tau\in\mathcal{C},\sigma\subset\tau} \tau$, so $\sigma \in \widehat{\mathcal{C}}$

A Cover for Intersection Complete Codes

Theorem (GIK)

$\text{code}(\text{cov}(\mathcal{C})) = \widehat{\mathcal{C}}$

- $\sigma \in \text{code}(\text{cov}(\mathcal{C})) \Rightarrow (\cap_{i\in\sigma} P(i)) \setminus \cup_{j\notin\sigma} P(j) \neq \varnothing$
- $j \in \bigcap_{\sigma\subset\tau}(\tau \setminus \sigma) \Rightarrow \bigcap_{i\in\sigma} P(i) \subset P(j)$
- Thus $\sigma = \displaystyle\bigcap_{\tau\in\mathcal{C},\sigma\subset\tau} \tau$, so $\sigma \in \widehat{\mathcal{C}}$

A Cover for Intersection Complete Codes

Theorem (GIK)

$\text{code}(\text{cov}(\mathcal{C})) = \widehat{\mathcal{C}}$

- $\sigma \in \widehat{\mathcal{C}} \Rightarrow \sigma = \sigma_1 \cap \cdots \cap \sigma_k, \sigma_i \in \mathcal{C}$
- $\frac{1}{k}\sum_{\ell=1}^{k} e_{\sigma_i} \in (\cap_{i\in\sigma} P(i)) \setminus \cup_{j\notin\sigma} P(j)$
- So $\sigma \in \text{code}(\text{cov}(\mathcal{C}))$.

A Cover for Intersection Complete Codes

Theorem (GIK)

$\text{code}(\text{cov}(\mathcal{C})) = \widehat{\mathcal{C}}$

- $\sigma \in \widehat{\mathcal{C}} \Rightarrow \sigma = \sigma_1 \cap \cdots \cap \sigma_k, \sigma_i \in \mathcal{C}$
- $\frac{1}{k} \sum_{\ell=1}^{k} e_{\sigma_i} \in (\cap_{i \in \sigma} P(i)) \setminus \cup_{j \notin \sigma} P(j)$
- So $\sigma \in \text{code}(\text{cov}(\mathcal{C}))$.

A Cover for Intersection Complete Codes

Theorem (GIK)

$\text{code}(\text{cov}(\mathcal{C})) = \widehat{\mathcal{C}}$

- $\sigma \in \widehat{\mathcal{C}} \Rightarrow \sigma = \sigma_1 \cap \cdots \cap \sigma_k, \sigma_i \in \mathcal{C}$
- $\frac{1}{k}\sum_{\ell=1}^{k} e_{\sigma_i} \in (\cap_{i\in\sigma} P(i)) \setminus \cup_{j\notin\sigma} P(j)$
- So $\sigma \in \text{code}(\text{cov}(\mathcal{C}))$.

A Cover for Intersection Complete Codes

Theorem (GIK)

$\text{code}(\text{cov}(\mathcal{C})) = \widehat{\mathcal{C}}$

- $\sigma \in \widehat{\mathcal{C}} \Rightarrow \sigma = \sigma_1 \cap \cdots \cap \sigma_k, \sigma_i \in \mathcal{C}$
- $\frac{1}{k} \sum_{\ell=1}^{k} e_{\sigma_i} \in (\cap_{i \in \sigma} P(i)) \setminus \cup_{j \notin \sigma} P(j)$
- So $\sigma \in \text{code}(\text{cov}(\mathcal{C}))$.

Intersection Violators

An **intersection violator** of a code $\mathcal{C}$ is a codeword $\sigma \in \widehat{\mathcal{C}} \setminus \mathcal{C}$.

By the theorem, intersection violators are the only possible obstructions to convexity.

Theorem (GIK)

For a certain large class of codes codes, the cover $\mathrm{cov}(\mathcal{C})$ can be "corrected" to produce a convex cover for $\mathcal{C}$.

Intersection Violators

An **intersection violator** of a code $\mathcal{C}$ is a codeword $\sigma \in \widehat{\mathcal{C}} \setminus \mathcal{C}$.

By the theorem, intersection violators are the only possible obstructions to convexity.

Theorem (GIK)

For a certain large class of codes codes, the cover $\text{cov}(\mathcal{C})$ can be "corrected" to produce a convex cover for $\mathcal{C}$.

Intersection Violators

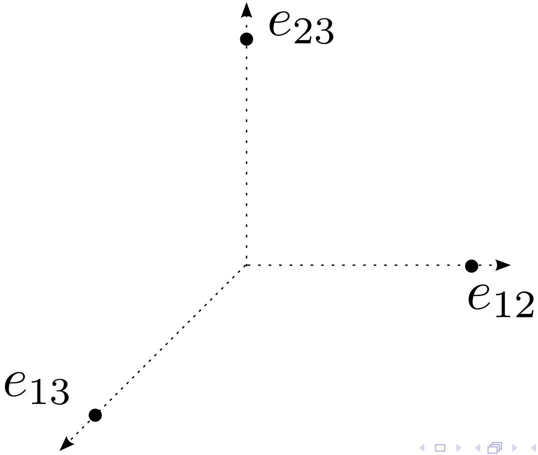
An **intersection violator** of a code $\mathcal{C}$ is a codeword $\sigma \in \widehat{\mathcal{C}} \setminus \mathcal{C}$. By the theorem, intersection violators are the only possible obstructions to convexity.

Theorem (GIK)

For a certain large class of codes codes, the cover $\text{cov}(\mathcal{C})$ can be "corrected" to produce a convex cover for $\mathcal{C}$.

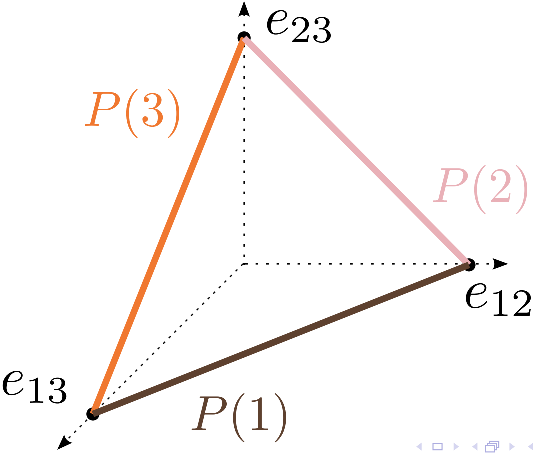
Example

$$\mathcal{C} = \{12, 13, 23, 123\}$$



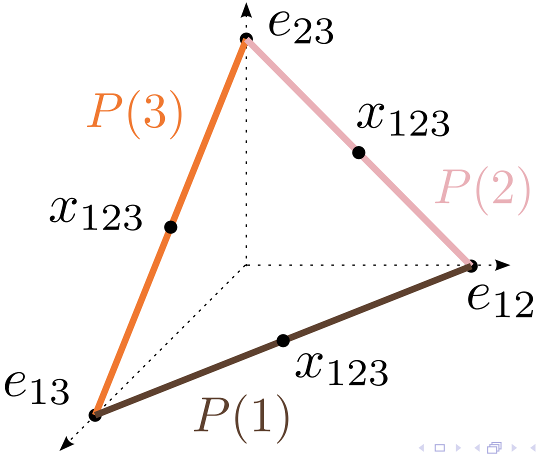
Example

$$\mathcal{C} = \{12, 13, 23, 123\}$$



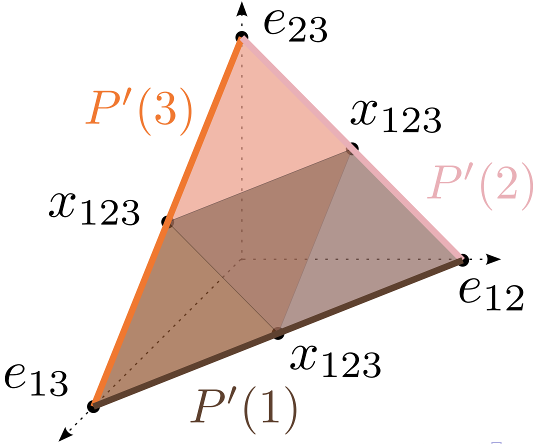
Example

$$\mathcal{C} = \{12, 13, 23, 123\}$$

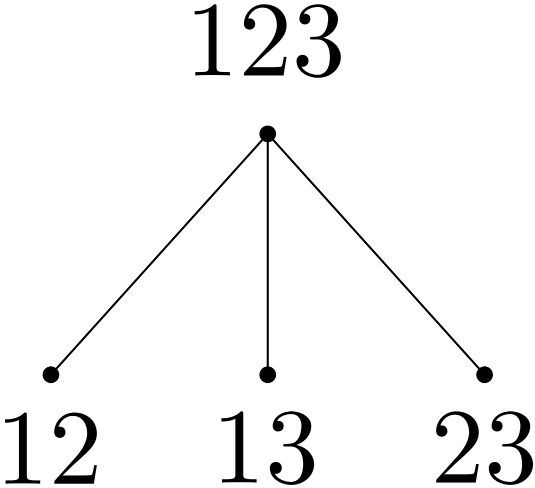


Example

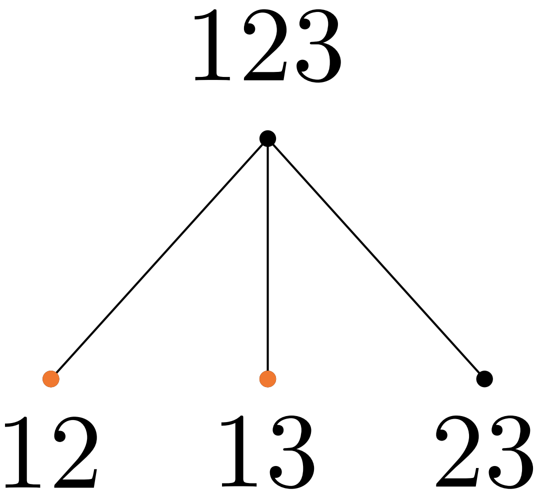
$$\mathcal{C} = \{12, 13, 23, 123\}$$



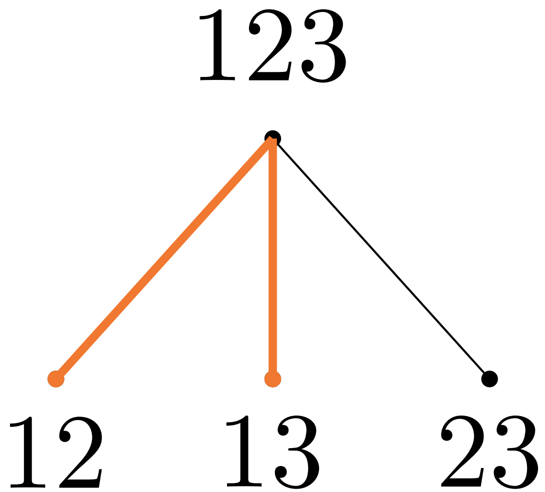
Example



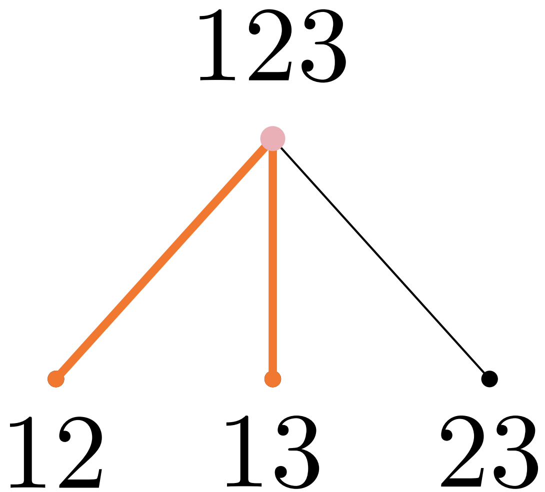
Example



Example



Example



Correcting the Cover

Theorem (GIK)

For a certain large class of codes codes, the cover $\text{cov}(\mathcal{C})$ can be "corrected" to produce a convex cover for $\mathcal{C}$.

Properties of the poset graph for the code determine whether or not our prescribed correction is feasible.

Correcting the Cover

Theorem (GIK)

For a certain large class of codes codes, the cover $\text{cov}(\mathcal{C})$ can be "corrected" to produce a convex cover for $\mathcal{C}$.

Properties of the poset graph for the code determine whether or not our prescribed correction is feasible.

Open/Closed Convex

- Every open convex code is closed convex.
  - Shrink each U_i by ε so that the intersection patterns are preserved.
  - (Details for finding ε are technical.)
  - Take the closure of the reulting sets.
- Not every closed convex code is open convex.
- A. Shiu proved that $\mathcal{C} = \{3, 4, 13, 14, 23, 34, 45, 123, 134, 145, 2345\}$ is closed convex, but not open convex.
- Open question: How can we detect when a code is open covex? Closed convex?

Open/Closed Convex

- Every open convex code is closed convex.
  - Shrink each U_i by ε so that the intersection patterns are preserved.
  - (Details for finding ε are technical.)
  - Take the closure of the reulting sets.
- Not every closed convex code is open convex.
- A. Shiu proved that $\mathcal{C} = \{3, 4, 13, 14, 23, 34, 45, 123, 134, 145, 2345\}$ is closed convex, but not open convex.
- Open question: How can we detect when a code is open covex? Closed convex?

Open/Closed Convex

- Every open convex code is closed convex.
    - Shrink each U_i by ε so that the intersection patterns are preserved.
    - (Details for finding ε are technical.)
    - Take the closure of the reulting sets.
- Not every closed convex code is open convex.
- A. Shiu proved that $\mathcal{C} = \{3, 4, 13, 14, 23, 34, 45, 123, 134, 145, 2345\}$ is closed convex, but not open convex.
- Open question: How can we detect when a code is open covex? Closed convex?

Open/Closed Convex

- Every open convex code is closed convex.
  - Shrink each U_i by ε so that the intersection patterns are preserved.
  - (Details for finding ε are technical.)
  - Take the closure of the reulting sets.
- Not every closed convex code is open convex.
- A. Shiu proved that $\mathcal{C} = \{3, 4, 13, 14, 23, 34, 45, 123, 134, 145, 2345\}$ is closed convex, but not open convex.
- Open question: How can we detect when a code is open covex? Closed convex?

Open/Closed Convex

- Every open convex code is closed convex.
  - Shrink each U_i by ε so that the intersection patterns are preserved.
  - (Details for finding ε are technical.)
  - Take the closure of the reulting sets.
- Not every closed convex code is open convex.
- A. Shiu proved that $\mathcal{C} = \{3, 4, 13, 14, 23, 34, 45, 123, 134, 145, 2345\}$ is closed convex, but not open convex.
- Open question: How can we detect when a code is open covex? Closed convex?

Open/Closed Convex

- Every open convex code is closed convex.
  - Shrink each U_i by ε so that the intersection patterns are preserved.
  - (Details for finding ε are technical.)
  - Take the closure of the reulting sets.
- Not every closed convex code is open convex.
- A. Shiu proved that $\mathcal{C} = \{3, 4, 13, 14, 23, 34, 45, 123, 134, 145, 2345\}$ is closed convex, but not open convex.
- Open question: How can we detect when a code is open covex? Closed convex?

Open/Closed Convex

- Every open convex code is closed convex.
  - Shrink each U_i by ε so that the intersection patterns are preserved.
  - (Details for finding ε are technical.)
  - Take the closure of the reulting sets.
- Not every closed convex code is open convex.
- A. Shiu proved that $\mathcal{C} = \{3, 4, 13, 14, 23, 34, 45, 123, 134, 145, 2345\}$ is closed convex, but not open convex.
- Open question: How can we detect when a code is open covex? Closed convex?

Dimension

- If d is the smallest positive integer for which $\mathcal{C}$ is the code of an open convex cover in $\mathbb{R}^d$, then d is the **open dimension** of $\mathcal{C}$.
- If d is the smallest positive integer for which $\mathcal{C}$ is the code of an closed convex cover in $\mathbb{R}^d$, then d is the **closed dimension** of $\mathcal{C}$.
- $\mathrm{opendim}(\mathcal{C}) \geq \mathrm{closeddim}(\mathcal{C})$
- If $\mathcal{C}$ is intersection complete or "correctable," then $\mathrm{closeddim}(\mathcal{C}) \leq |\mathcal{C}| - 1$.

Dimension

- If d is the smallest positive integer for which $\mathcal{C}$ is the code of an open convex cover in $\mathbb{R}^d$, then d is the **open dimension** of $\mathcal{C}$.
- If d is the smallest positive integer for which $\mathcal{C}$ is the code of an closed convex cover in $\mathbb{R}^d$, then d is the **closed dimension** of $\mathcal{C}$.
- $\mathrm{opendim}(\mathcal{C}) \geq \mathrm{closeddim}(\mathcal{C})$
- If $\mathcal{C}$ is intersection complete or "correctable," then $\mathrm{closeddim}(\mathcal{C}) \leq |\mathcal{C}| - 1$.

Dimension

- If d is the smallest positive integer for which $\mathcal{C}$ is the code of an open convex cover in $\mathbb{R}^d$, then d is the **open dimension** of $\mathcal{C}$.
- If d is the smallest positive integer for which $\mathcal{C}$ is the code of an closed convex cover in $\mathbb{R}^d$, then d is the **closed dimension** of $\mathcal{C}$.
- $\text{opendim}(\mathcal{C}) \geq \text{closeddim}(\mathcal{C})$
- If $\mathcal{C}$ is intersection complete or "correctable," then $\text{closeddim}(\mathcal{C}) \leq |\mathcal{C}| - 1$.

Dimension

- If d is the smallest positive integer for which $\mathcal{C}$ is the code of an open convex cover in $\mathbb{R}^d$, then d is the **open dimension** of $\mathcal{C}$.
- If d is the smallest positive integer for which $\mathcal{C}$ is the code of an closed convex cover in $\mathbb{R}^d$, then d is the **closed dimension** of $\mathcal{C}$.
- $\text{opendim}(\mathcal{C}) \geq \text{closeddim}(\mathcal{C})$
- If $\mathcal{C}$ is intersection complete or "correctable," then $\text{closeddim}(\mathcal{C}) \leq |\mathcal{C}| - 1$.

Dimension

- If d is the smallest positive integer for which $\mathcal{C}$ is the code of an open convex cover in $\mathbb{R}^d$, then d is the **open dimension** of $\mathcal{C}$.
- If d is the smallest positive integer for which $\mathcal{C}$ is the code of an closed convex cover in $\mathbb{R}^d$, then d is the **closed dimension** of $\mathcal{C}$.
- $\text{opendim}(\mathcal{C}) \geq \text{closeddim}(\mathcal{C})$
- If $\mathcal{C}$ is intersection complete or "correctable," then $\text{closeddim}(\mathcal{C}) \leq |\mathcal{C}| - 1$.

Example

- The code $\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$ is both open convex and closed convex.
- $\text{closeddim}(\mathcal{C}) = 2$
- $\text{opendim}(\mathcal{C}) = 3$

Example

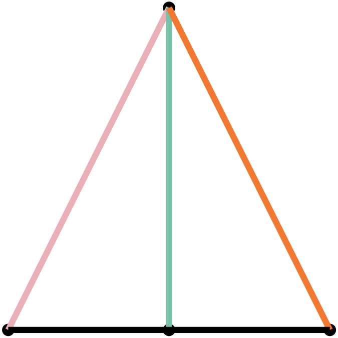
- The code $\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$ is both open convex and closed convex.
- $\text{closeddim}(\mathcal{C}) = 2$
- $\text{opendim}(\mathcal{C}) = 3$

Example

- The code $\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$ is both open convex and closed convex.
- $\text{closeddim}(\mathcal{C}) = 2$
- $\text{opendim}(\mathcal{C}) = 3$

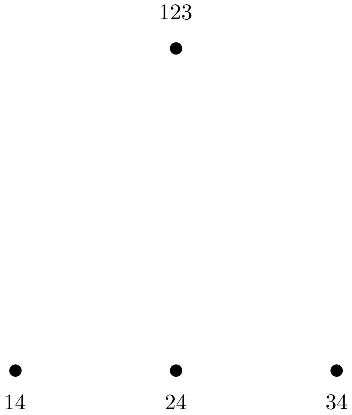
Closed Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



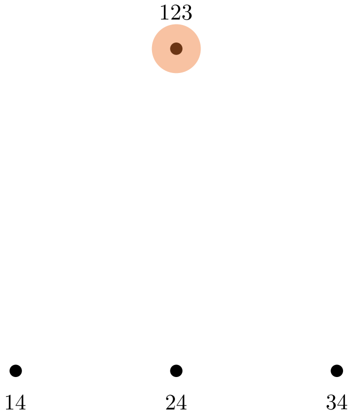
Open Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



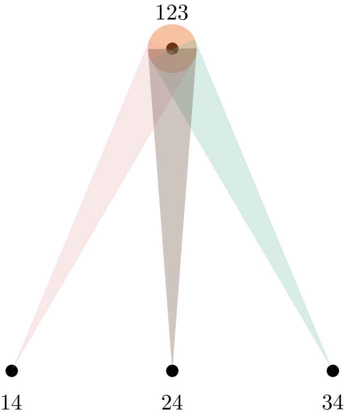
Open Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



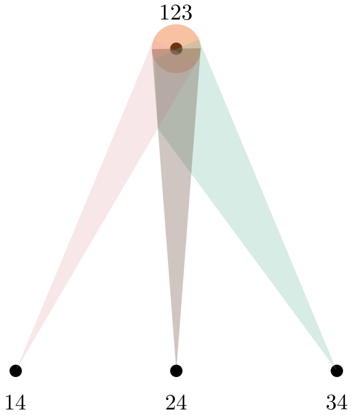
Open Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



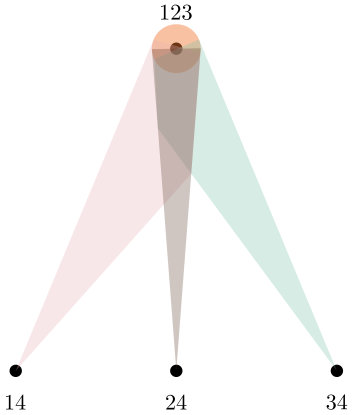
Open Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



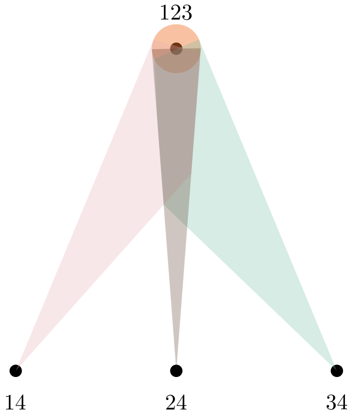
Open Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



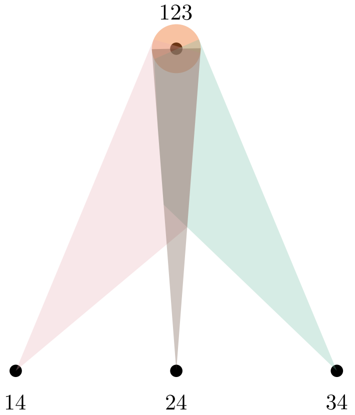
Open Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



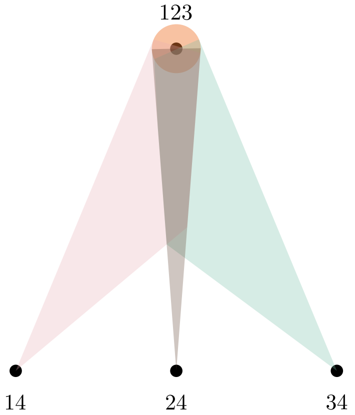
Open Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



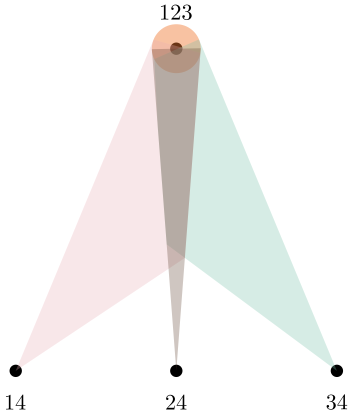
Open Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



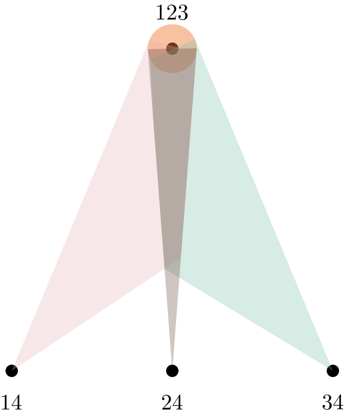
Open Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



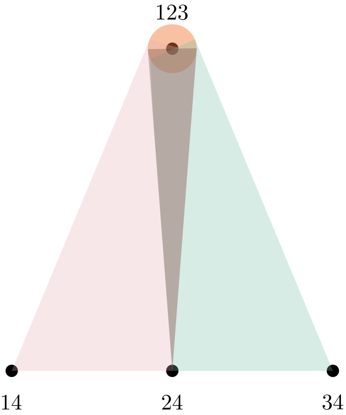
Open Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



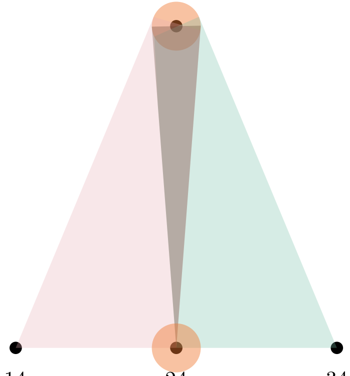
Open Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



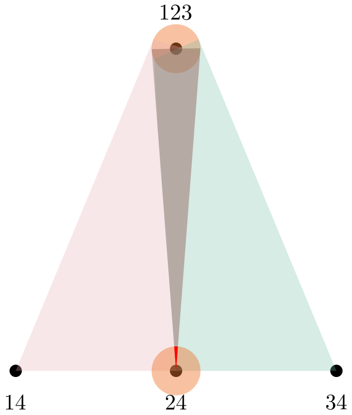
Open Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



Open Cover

$$\mathcal{C} = \{1, 2, 3, 4, 14, 24, 34, 123\}$$



Open Question

Given an open convex code $\mathcal{C}$, what are bounds on the open dimension?

Open Question

Given a closed convex code $\mathcal{C}$, what are bounds on the open dimension?

Open Question

Given an open convex code $\mathcal{C}$, what are bounds on the open dimension?

Open Question

Given a closed convex code $\mathcal{C}$, what are bounds on the open dimension?

References

- C. Curto, E. Gross, J. Jeffries, K. Morrison, M. Omar, Z. Rosen, A. Shiu, N. Youngs. *What makes a neural code convex?*, (pre-print 2015).
- C. Giusti, V. Itskov, –. *On convex codes an intersection violators* (in preparation).
- L. Lienkaemper. *When do neural codes come from convex or good covers?* (pre-print 2015).