MATH 280
Abstract Thinking
Workbook

Bill Kronholm

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Contents

Introduction

This document is a compilation of worksheets I have created for use in a one semester abstract thinking course. This course is a bridge course, introducing students to methods of proof and concepts which are frequently encountered in upper division mathematics courses. My course met three days a week for 50 minutes per class. The weekly routine was (roughly) worksheet, worksheet, worksheet. That is, I seldom lectured. Students would work on these in groups in class and present some of their work at the beginning of the following class. Below, if a worksheet seems to be assuming knowledge of content which has not been covered by previous worksheets, it is likely that this content was covered in lecture.

For my class, we used "The Art of the Proof" by Adrian Riskin and published with a Creative Commons license, and all section references below are to the version of this textbook to which I have made some edits to. In some cases, examples, exercises, and perhaps even prose from "The Art of the Proof" have been incorporated into these worksheets. Attribution to Riskin is due and hereby given. The version I used is linked to on my web page www.whittier.edu/facultypages/wkronholm/teaching.html. Of course, these worksheets can be useful in this study of mathematics in the absence of that, or any, textbook.

If you notice any typos or other errors, or are interested in my LaTeX code, please send me an email: wkronholm@whittier.edu. I would also appreciate hearing from instructors who use these materials in their courses.

I have also prepared solutions to most of the worksheets below. If you are using these worksheets as teaching materials and are interested in the solutions, email me and I will send them to you.

Chapter 1

Propositional Logic

1.1 Proofs and Propositions

1.1.1 Propositions

Sentences which have a truth value are called *propositions* or *statements*. (The two words "proposition" and "statement" are synonymous.) That is, a sentence is a proposition if either it is true or it is false. Notice we do not need to know in advance the truth value of a sentence in order to determine whether or not it is a proposition. Sentences which are subjective (i.e. Picasso was the best painter ever) are not statements.

Exercise 1.1.1. Determine which of the following sentences are propositions. You do not need to determine the truth value of any of the statements. (Although, you may, if you wish.)

(a) Bees are insects.

(b) All insects have four legs.

(c) The number four is even.

(d) The number four is odd.

(e) The number four is green.

(f) Green is a color.

(g) Green is a good color.

(h) My favorite color is orange.

(i) There is life on Mars.

(j) Every time Barack Obama has been on the moon he has not needed a spacesuit.

(k) This sentence is false.

(l) This sentence is true.

(m) Cats are cats.

(n) Every student in this class will get a grade of B.

(o) The American Civil War was senseless.

(p) The previous sentence is a proposition.

(q) I feel fine.

(r) All of this makes sense.

Exercise 1.1.2. Write three sentences which are propositions and three which are not. None of these sentences can be ones which appear anywhere else in these pages.

1.1.2 Propositional Logic

Just as in algebra we use letters to represent numbers (or functions, or operators, or ...), we will use letters to stand for propositions. It is traditional to use capital Roman letters beginning with P. So, if we have a list of propositions, they could be P, Q, R, etc. Or maybe we have a bigger list of propositions P_1, P_2, P_3, ... We'll declare a proposition by writing something like:

Let P be the proposition "Homework is due on Tuesdays."

Negation

Suppose P is a proposition. Then the sentence "P is false" is also a proposition called the *negation* of P. The negation of P is denoted $\neg P$. For example, the negation of "Homework is due on Tuesdays" is " 'Homework is due on Tuesdays' is false." Notice that this makes for some awkward English sentences.

Exercise 1.1.3. Explain why "P is false" is also a proposition whenever P is.

Exercise 1.1.4. Choose a proposition from Exercise 1.1.1 and call it P.

(a) Write P in English in the space below.

(b) Write $\neg P$ in English in the space below. Be sure to use precisely the definition of negation given above.

(c) Write $\neg\neg P$ in English in the space below. Again, be sure to use precisely the definition of negation given above.

(d) Rewrite both $\neg P$ and $\neg\neg P$ in more elegant, grammatically correct English.

Whenever you are writing, you want to use elegant, grammatically correct English.

Conjunction

If we have two statements P and Q, then we can make a new statement "P and Q" called the *conjunction* of P and Q denoted $P \wedge Q$. The conjunction is of P and Q is equivalent to the statement "P is true, and also Q is true." Or put another way: "Not only is P true, but Q is true, too."

Exercise 1.1.5. Choose two statements from Exercise 1.1.1 and call one of them P and the other Q.

(a) Write out both P and Q below.

(b) Write the statement $P \wedge Q$ below. Choose a phrasing which is equivalent to the logical statement $P \wedge Q$ but which is not awkward. (e.g. Instead of "Ralph is a cat and Frank is a cat," you would want to write "Ralph and Frank are cats.")

Disjunction

If we have two statements P and Q, then we can make a new statement "P or Q" called the *disjunction* of P and Q denoted $P \vee Q$. The disjunction is of P and Q is equivalent to the statement "P is true or Q is true." We have to be careful here. In conversational English, the usage of "or" is typically in an exclusive sense. That is, when someone says, "I'm either going to Whittier College or going to live in a tent under the freeway," we don't expect them to end up doing both. However, in mathematics we allow for this possibility when discussing "or" statements. For example, the statement "Chemistry is a discipline in the sciences or English is a discipline in the humanities" is a true statement. The statement "Chemistry is a discipline in the sciences or English is a discipline in the sciences" is also a true statement.

Exercise 1.1.6. Choose two statements from Exercise 1.1.1 and call one of them R and the other S.

(a) Write out both R and S below.

(b) Write the statement $R \vee S$ below. Choose a phrasing which is equivalent to the logical statement $R \vee S$ but which is not awkward.

Complex Statements

We now have three operations we can perform on statements: negation, conjunction, and disjunction. We can make more complicated statements by using two or more of these operations. However, we need to be careful.

Exercise 1.1.7. Choose statements P, Q, and R from Exercise 1.1.1.

(a) Write out P, Q, and R below.

(b) Write an elegant English sentence corresponding to the statement $P \wedge (Q \vee R)$.

(c) Write an elegant English sentence corresponding to the statement $(P \wedge Q) \vee R$.

(d) Reflect on the two sentences you wrote. Do they capture the same idea? (*Hint:* They shouldn't.)

(e) Write an elegant English sentence corresponding to the statement $\neg(Q \vee R)$.

(f) Write an elegant English sentence corresponding to the statement $(\neg Q) \vee (\neg R)$.

(g) Reflect on the two sentences you wrote. Do they capture the same idea? (*Hint:* They shouldn't.)

1.2 Truth Tables and De Morgan's Laws

1.2.1 Truth Tables

Suppose we have some propositions, say P, Q, and R. As you saw last time, we can combine these together using the operations of $\neg$, $\vee$, and $\wedge$ to get new propositions like $Q \wedge R$, $\neg R \vee P$, $(P \wedge Q) \vee \neg R$, etc. It would be nice to know the truth values of these new propositions. Of course, the truth value of a proposition like $(P \wedge Q) \vee \neg R$ will certainly depend on the truth values of P, Q, and R. One tool for helping us keep track of this is something called a *truth table*.

The simplest example of a truth table is that for the negation of a single proposition P. It looks like this:

P	$\neg P$
F	T
T	F

The possible truth values for P are in the first column. Once you pick a truth value for P, the truth value for $\neg P$ is determined and entered in the column below $\neg P$. Explicitly, this table tells us that if P is true, then $\neg P$ is false. It also tells us that if P is false, then $\neg P$ is true.

Here are the truth tables for $P \wedge Q$ and $P \vee Q$.

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

Notice that the only case in which the statement $P \wedge Q$ is true is when both P and Q are true. Similarly, $P \vee Q$ is only false when both P and Q are false.

These three truth tables are enough to build truth tables for more complicated statements using any number of propositions and any number of $\neg$'s, $\vee$'s, and $\wedge$'s.

Exercise 1.2.1. Build a truth table for $(P \wedge Q) \wedge R$. I've started it for you. (*Hint*: Start by making a column for $P \wedge Q$ and use that to help you get the column for $(P \wedge Q) \wedge R$.)

P	Q	R			
F	F	F			
F	F	T			
F	T	F			
F	T	T			
T	F	F			
T	F	T			
T	T	F			
T	T	T			

Exercise 1.2.2. Build a truth table for $((\neg P) \vee Q) \wedge Q$.

Exercise 1.2.3. Build a truth table for $\neg(P \vee Q)$

Exercise 1.2.4. Build a truth table for $\neg P \wedge \neg Q$. (You should interpret this statement as $(\neg P) \wedge (\neg Q)$.)

Exercise 1.2.5. Build a truth table for $\neg(P \wedge Q)$

Exercise 1.2.6. Build a truth table for $\neg P \vee \neg Q$. (You should interpret this statement as $(\neg P) \vee (\neg Q)$.)

Exercise 1.2.7. Build a truth table for $(P \wedge (Q \vee \neg R)) \vee P$.

Exercise 1.2.8. Make up your own complex statement involving P, Q, and R. Make sure that you use each of the symbols $\neg$, $\wedge$, and $\vee$ at least once. Build a truth table for your statement.

1.2.2 De Morgan's Laws

Compare the column you created for $\neg(P \vee Q)$ in Exercise 1.2.3 with the one you created for $\neg P \wedge \neg Q$ in Exercise 1.2.4. Do you notice something?

Exercise 1.2.9. What do you notice?

Now compare the column you created for $\neg(P \wedge Q)$ in Exercise 1.2.5 with the one you created for $\neg P \vee \neg Q$ in Exercise 1.2.6. Do you notice something?

Exercise 1.2.10. What do you notice?

What you should have noticed is that the entries in the truth tables for $\neg(P \vee Q)$ and those for $\neg P \wedge \neg Q$ are exactly the same. That is, regardless of the individual truth values of P and Q, the truth values of $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ will always be the same. So, these propositions are equivalent in the sense that they capture the same idea. Similarly, the propositions $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are equivalent. These two equivalences are what are known as *De Morgan's Laws* and are summarized below.

Definition 1.2.11. De Morgan's Laws are the logival equivalences below.

$$\neg(P \vee Q) \text{ is equivalent to } \neg P \wedge \neg Q$$
$$\neg(P \wedge Q) \text{ is equivalent to } \neg P \vee \neg Q$$

These rules can help to simplify compound logical expressions. There are others that can help as well.

Exercise 1.2.12. Construct truth tables for $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$. Deduce a distributive law.

Exercise 1.2.13. Construct truth tables for $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$. Deduce a distributive law.

Exercise 1.2.14. Consider the statement a parent makes to a child: “Either clean up your room or you are grounded!” After hearing this, the child cleans her room. After which the parent replies, “Good. You’re grounded.” Has the child been done an injustice? Explain.

Exercise 1.2.15. Consider the following quote from a Daffy Duck cartoon[1]:

> “There’s something awfully screwy about this fight, or my name isn’t Lattimore... and it isn’t.”

Is there something awfully screwy about this fight? Explain.

[1] “To Duck or not to Duck”, Warner Bros., 1943. Directed by Chuck Jones.

1.3 Implication, Converse, Contrapositive, Logical Equivalence, Tautology, and Contradiction

Previously you have studied the propositional operations of $\neg$, $\vee$, and $\wedge$. We have two more such operators to explore: $\rightarrow$ and $\leftrightarrow$.

1.3.1 Implication

The statement $P \rightarrow Q$ is read as "P implies Q" or "If P, then Q." The intent of the implication is this: If P is true, and the implication $P \rightarrow Q$ is true, then we can conclude that Q is true. Implications are probably the most common form the statement of a mathematical theorem takes, so they are worth discussing in some detail.

Consider the statement "If my pet is a cat, then my pet is a mammal." If we let P be the statement "My pet is a cat" and Q be the statement "My pet is a mammal," then the given statement is of the form $P \rightarrow Q$. What would it take for $P \rightarrow Q$ to be true? Maybe it's better to start with the negation.

Exercise 1.3.1. What would need to be true for the statement "If my pet is a cat, then my pet is a mammal" to be false?

Did you say that your pet would have to be a cat and your pet would also not be a mammal? Good. That's the statement $P \wedge \neg Q$.

Exercise 1.3.2. Make a truth table for the statement $P \wedge \neg Q$.

Since $\neg(P \rightarrow Q)$ is the same as $P \wedge \neg Q$, it must be that $P \rightarrow Q$ is the same as $\neg P \vee Q$.

Exercise 1.3.3. Extend your truth table above to have a column for $P \rightarrow Q$.

Here is the truth table:

P	Q	$P \to Q$
F	F	T
F	T	T
T	F	F
T	T	T

Exercise 1.3.4. Construct a truth table for the statement $Q \wedge (P \to (\neg R \vee Q))$.

Exercise 1.3.5. Construct a truth table for the statement $\neg(P \wedge Q) \to R$.

Definition 1.3.6. In the implication $P \to Q$, the statement$P$ is the **antecedent** (or **hypothesis**) and the statement Q is the **consequent** (or **conclusion**).

Exercise 1.3.7. Determine the antecedent and consequent of each statement below. It might be helpful to rephrase the statement in the form "If P, then Q."

(a) If you don't clean your room, you'll be grounded.

(b) Those who drink and drive go to jail.

(c) All odd numbers are prime.

(d) The students who study do well on the exam.

(e) You do well on exams because you study.

(f) The coin came up heads. Therefore I win the bet.

1.3.2 Converse

The **converse** to the implication $P \to Q$ is the implication $Q \to P$. While the statements $P \wedge Q$ and $Q \wedge P$ capture the same idea, as do $P \vee Q$ and $Q \vee P$, this is not the case for the statements $P \to Q$ and $Q \to P$. You can see this by constructing truth tables for the two statements.

Exercise 1.3.8. Construct a truth table for the statements $P \to Q$ and $Q \to P$. Observe that the columns for $P \to Q$ and $Q \to P$ are different.

Of course, you don't need a truth table to convince you of this.

Exercise 1.3.9. Write in English statements of the form $P \to Q$ so that

(a) your statement is false, but the converse is true.

(b) your statement is true, but the converse is false.

(c) your statement is true, and the converse is also true.

(d) your statement is false, and the converse is also false.

1.3.3 Contrapositive

The **contrapositive** to the implication $P \to Q$ is the implication $\neg Q \to \neg P$.

Exercise 1.3.10. Construct a truth table for $\neg Q \to \neg P$. Compare it to the truth table for $P \to Q$. What do you notice?

Exercise 1.3.11. Write the contrapositive to each of the statements from Exercise 1.3.7.

When we get to writing proofs, it will occasionally be more advantageous to prove the contrapositive to the implication to be proven. This is OK because the contrapositive to an implication is essentially the same as the original implication.

1.3.4 Logical Equivalence

All of this talk about statements being "essentially the same" needs some nailing down. We say that two propositions are **logically equivalent** if they have the same truth value for every possible combinations of truth values of their component propositions. That is, two statements are logically equivalent if the entries in the truth tables for the two statements are exactly the same. If P and Q are logically equivalent, then we write this as $P \leftrightarrow Q$. This is often read as "P is equivalent to Q" or "P if and only if Q." A truth table for $P \leftrightarrow Q$ is below.

P	Q	$P \leftrightarrow Q$
F	F	T
F	T	F
T	F	F
T	T	T

Exercise 1.3.12. Comprise a list of all statements in the previous pages which reference logical equivalences. For example, in Exercise 1.3.2 you determined that $\neg(P \to Q) \leftrightarrow (P \wedge \neg Q)$.

Exercise 1.3.13. Build a truth table for $\neg(P \to Q) \leftrightarrow (P \wedge \neg Q)$. What do you notice?

1.3.5 Tautology and Contradiction

Any statement which is true regardless of the truth value of its component propositions is called a **tautology**. Any statement which is false regardless of the truth value of its component propositions is called a **contradiction**. In truth tables, we can easily identify statements that are tautologies because these statements have only T's in their columns. Similarly, statements which are contradictions have only F's in their columns.

Exercise 1.3.14. Show that the statement $P \vee \neg P$ is a tautology and that $P \wedge \neg P$ is a contradiction.

The previous exercise really is just evidence that we are working in a consistent logical system. That is, no statement which we consider can be both true and false, nor can any statement fail to be either true or false. This is the type of system that most mathematicians like to work in, but there are some out there, called **constructivists**, who allow for the possibility that a statement be neither true nor false. For the purposes of this course, we are not constructivists.

Perhaps it is not surprising, but all of the usual rules of deduction, when properly stated in the logical system we have been describing, are tautologies.

Exercise 1.3.15. Verify that the following are tautologies.

(a) (Conjunction elimination) $(P \wedge Q) \to P$

(b) (Disjunction introduction) $P \to (P \vee Q)$

(c) (Double negative elimination) $\neg\neg P \to P$

(d) (Modus ponens) $(P \wedge (P \to Q)) \to Q$

(e) (Modus tollens) $((P \to Q) \wedge \neg Q) \to \neg P$

(f) (Biconditional introduction) $((P \to Q) \wedge (Q \to P)) \to (P \leftrightarrow Q)$

(g) (Disjunction elimination) $((P \vee Q) \wedge (P \to R) \wedge (Q \to R)) \to R$

Chapter 2

Sets

2.1 Sets

This worksheet discusses material corresponding roughly to sections 3.1-3.2 of your textbook.

2.1.1 Preliminaries

Sets are the first mathematical objects which we will write proofs about. Before we can do that, it's good to get acquainted with these mathematical objects called sets.

Unfortunately, the best we can do for a definition of a set is the following:

Definition 2.1.1. A **set** is a collection A of distinct objects considered as an object in its own right. If x is one of the distinct objects in the set A, then we denote this by $x \in A$ and say that x is a **member** of the set A. If x is not a member of A, then we write $x \notin A$.

Of course, this begs the question of what a collection is and what an object is, but we have to ignore that. We have a couple of ways of describing a set. One way is by specifying a rule by which membership can be determined.

- A is the set of colors in the American flag.
- B is the set of animals in the San Diego Zoo.
- C is the set of tools in the set design and construction studio in the Shannon Center.
- D is the set of even positive integers.

Another way we can describe a set is by listing all of the members of the set inside of curly braces, which are these things: { }. This might be feasible for sets with finitely many members, but is probably unreasonable for sets with infinitely many members. We have some workarounds for this, however. Here are some examples.

- $E = \{1, 2, 3, 4\}$
- $F = \{\text{red, white, blue}\}$
- $G = \{1, 2, 3, \ldots, 250\}$
- $H = \{1, 2, 3, \ldots\}$
- $I = \{2, 4, 6, \ldots\}$

In the above, the ellipsis $(\ldots)$ indicates that the obvious pattern continues. One thing to notice is that set membership provides us with statements. That is given any object x and any set A, then the sentence "$x \in A$" is either a true statement, or else it's a false statement. So, for example, $300 \in H$ and $300 \in I$ are true statements, but $300 \in G$ is false.

If we're going to prove anything about sets, then we need something proof worthy to talk about. That is, we need some definitions.

Definition 2.1.2. A set A is a **subset** of a set B if the statement $(x \in A) \rightarrow (x \in B)$ is true for every object x. If A is a subset of B, then we write $A \subset B$ or $A \subseteq B$.[1]

Exercise 2.1.3. Determine which of the sets A, B, C, D, E, F, G, H, and I from the previous page are subsets of the others.

Exercise 2.1.4. Negate the definition of subset. That is, determine conditions under which a set A is *not* a subset of a set B. (In this case, we would write $A \not\subset B$ or $A \nsubseteq B$.)

Definition 2.1.5. If A and B are sets, then we say that A and B are **equal** if $A \subseteq B$ and $B \subseteq A$. In this case, we write $A = B$.

Exercise 2.1.6. Determine which of the sets A, B, C, D, E, F, G, H, and I from the previous page are equal.

Exercise 2.1.7. Negate the definition of equality. That is, determine conditions under which a set A is *not* equal to a set B. (Perhaps not surprisingly, in this case we write $A \neq B$.)

[1] More on this choice of notation later. Suffice it to say for now that it varies from author to author.

We also have the so-called set-builder notation for describing sets. For example, the notation $A = \{x \mid x \text{ is a real number and } x^2 = x\}$ means the following: A is a set whose elements are the real numbers which satisfy the equation $x^2 = x$. The vertical bar | is read as "such that." [2] Hopefully, you can see that $A = \{0, 1\}$.

We have some special sets.

Definition 2.1.8. The following are sets:

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, \dots\} \\
\mathbb{Z} &= \{\dots, -2, -1, 0, 1, 2, \dots\} \\
\mathbb{Q} &= \{r \mid r = \frac{p}{q} \text{ for some } p, q \in \mathbb{Z} \text{ and } q \neq 0\} \\
\mathbb{R} &= \{x \mid x \text{ is a real number}\}
\end{aligned}$$

These sets are referenced often enough that they deserve the special status afforded them by fixed notation. That is, while the set A may refer to different sets at different times, the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ will always[3] mean the sets defined above.

Exercise 2.1.9. Determine explicitly the members of the sets below.

(a) $A = \{x \in \mathbb{N} \mid (2x - 4)(3x + 1)(x + 1)(x - \pi) = 0\}$

(b) $B = \{x \in \mathbb{Z} \mid (2x - 4)(3x + 1)(x + 1)(x - \pi) = 0\}$

(c) $C = \{x \in \mathbb{Q} \mid (2x - 4)(3x + 1)(x + 1)(x - \pi) = 0\}$

(d) $D = \{x \in \mathbb{R} \mid (2x - 4)(3x + 1)(x + 1)(x - \pi) = 0\}$

[2] Often a colon, : , is used in place of the bar, |. This is another example of an instance where there is less than uniform agreement for notations. It's good to be aware that there are other conventions.

[3] Almost. The set $\mathbb{N}$ will either refer to $\{0, 1, 2, 3, \dots\}$ or the set $\{1, 2, 3, \dots\}$, depending on which is more convenient at the time.

2.1.2 Union and Intersection

The first two basic operations we will perform on sets are union and intersection.

Definition 2.1.10. If A and B are sets, then the **union** of A and B is the set $A \cup B$ which has members determined by

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

Definition 2.1.11. If A and B are sets, then the **intersection** of A and B is the set $A \cap B$ which has members determined by

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

Exercise 2.1.12. Consider the sets below.

$$\begin{aligned} A &= \{1,2,3,4\} \\ B &= \{2,4,6\} \\ C &= \{1,3,5\} \end{aligned}$$

Compute:

(a) $A \cup B$
(b) $A \cup C$
(c) $B \cup C$
(d) $A \cap B$
(e) $A \cap C$
(f) $B \cap C$

Hopefully, you noticed something funny with the last one. It seems like B and C have no members in common, and they don't. We still want to be able to make sense of the intersection, however. This motivates us to make a definition.

Definition 2.1.13. The set which has no members is called the **empty set** and is denoted $\varnothing$.

The empty set is maybe a little strange, but the statement $x \in \varnothing$ is *false*, regardless of what x is.

Exercise 2.1.14. Consider the same sets A, B, and C as in the previous exercise. Compute:

(a) $A \cup \varnothing$
(b) $B \cup \varnothing$
(c) $C \cup \varnothing$
(d) $A \cap \varnothing$
(e) $B \cap \varnothing$
(f) $C \cap \varnothing$

Exercise 2.1.15. Determine the truth value of the statements below.

(a) $x \in \varnothing \to x \in \mathbb{R}$.
(b) $x \in \varnothing \to x$ is on the moon.
(c) $\varnothing \subseteq \mathbb{N}$
(d) $\varnothing \subseteq \mathbb{Z}$
(e) $\varnothing \subseteq \mathbb{Q}$
(f) $\varnothing \subseteq \mathbb{R}$

2.2 Proving Theorems about Sets

This worksheet discusses material corresponding roughly to sections 3.4-3.7 of your textbook.

Recall that one basic strategy in beginning a proof is to rephrase the statement to be proven as an implication.

Exercise 2.2.1. Prove: $A \cap (B \cup C) \subset (A \cap C) \cup (A \cap B)$.

Exercise 2.2.2. Prove: $(A \cap C) \cup (A \cap B) \subset A \cap (B \cup C)$.

Exercise 2.2.3. Prove: $A \cap (B \cup C) = (A \cap C) \cup (A \cap B)$.

While examples cannot prove a statement to be true, they can be used to prove that a statement is false. Such an example is called a **counterexample**.

Exercise 2.2.4. Consider the statement $A \subset A \cap B$. While this statement may be true for some specific sets A and B, the statement is in general false. Produce a counterexample for this statement. That is, give an example of a set A and a set B so that $A \not\subset A \cap B$. Explain why your set is a counterexample.

Exercise 2.2.5. Prove or find a counterexample: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Exercise 2.2.6. Prove or find a counterexample: $A \cup (B \cap C) \subset (A \cup B) \cap C$.

Exercise 2.2.7. Prove or find a counterexample: If $A \subset B$ and $C \subset D$, then $A \cup C \subset B \cup D$.

Exercise 2.2.8. Prove or find a counterexample: If $A \subset B$ and $C \subset D$, then $A \cap C \subset B \cap D$.

Exercise 2.2.9. Prove or find a counterexample: If $A \subset B$ then $A \cup B = A$.

Exercise 2.2.10. Prove or find a counterexample: If $A \subset B$ then $A \cap B = A$.

Exercise 2.2.11. Prove or find a counterexample: If $A \subset B$ then $A \cup B = B$.

2.3 More on Sets

This worksheet discusses material corresponding roughly to sections 3.6-3.7 of your textbook.

Definition 2.3.1. Let A and B be sets. Then the **difference** of A and B is the set $A - B$ of all elements of A which are not elements of B. That is,

$$A - B = \{x \in A \mid x \notin B\}$$

Exercise 2.3.2. For each pair of sets below, compute $A - B$ and $B - A$.

(a) $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8, 9\}$

(b) $A = \{2, 3, 5, 7, 11, 13, \dots\}$, $B = \{2, 4, 6, 8, \dots\}$

(c) $A = \mathbb{N}$, $B = \mathbb{Z}$

(d) $A = \{x \mid x = 2n + 1 \text{ for some } n \in \mathbb{N}\}$, $B = \varnothing$

Exercise 2.3.3. Prove: $A - (B \cup C) = (A - B) \cap (A - C)$.

Exercise 2.3.4. Prove or disprove: $A - B = B - A$.

Exercise 2.3.5. Prove or disprove: If $A - B = \varnothing$, then $A = B$.

Exercise 2.3.6. Prove or disprove: $A - (B - C) = A - (B \cup C)$.

Exercise 2.3.7. Prove or disprove: $(A - B) \cap (B - A) = \varnothing$.

Exercise 2.3.8. Prove or disprove: $A - (B \cap C) = (A - B) \cup (A - C)$.

Exercise 2.3.9. Prove or disprove: $A - (B - C) = (A - B) - C$.

Exercise 2.3.10. Prove or disprove: $A - (B - C) = (A - B) \cup C$.

Chapter 3

Quantifiers

3.1 Quantifiers

This worksheet discusses material corresponding roughly to sections 4.1 of your textbook.

3.1.1 Open Sentences

Briefly, an **open sentence** is one with a variable. For example, the sentence "x is a real number." is an open sentence with **free variable** x. Notice this sentence is not a proposition, since any truth value would be dependent upon what x is referring to. If we denote this sentence by $P(x)$, then we obtain a proposition by "plugging in" some value for x. For example, the sentence $P(\pi)$ translates to "π is a real number," which is a (true) statement. Also, P(Prof. Kronholm) is the sentence "Prof. Kronholm is a real number," which is also a statement (which happens to be false).

Exercise 3.1.1. Determine the free variable in each open sentence below.

(a) n is prime.

(b) T is a letter of the Greek alphabet.

(c) y is a chromosome.

Exercise 3.1.2. Label the open sentences above as $A(n)$, $B(T)$, and $C(y)$, respectively. Write out English sentences for the statements $A(15)$, $B(\pi)$, $C(42)$, A(Leonardo), B(Leonardo), and C(Leonardo).

3.1.2 Universal Quantifier

One way to make an open sentence into a statement is to use the **universal quantifier** "for every." If we take the example sentence "x is a real number" and apply the universal quantifier to it, we obtain the statement "For every x, x is a real number," or "For each x, x is a real number," or even "Every x is a real number." In symbols, we write $(\forall x)P(x)$. Strictly speaking, this is still vague, since the scope of the free variable x is not defined. We can specify the scope like so:

$$(\forall x \in \mathbb{N})P(x)$$

The scope of the variable x is determined by the notation $x \in \mathbb{N}$.

Exercise 3.1.3. Let $P(n)$ be the sentence "n is odd," $Q(n)$ be the sentence "n is greater than 4," and $R(n)$ be the sentence "n is negative." Translate the following symbols into elegant English sentences.

(a) $(\forall n \in \mathbb{N})P(n)$.

(b) $(\forall n \in \mathbb{Z})(\neg Q(n))$.

(c) $(\forall n \in \mathbb{N})(R(n) \vee P(n))$.

(d) $(\forall n \in \mathbb{Z})(Q(n) \wedge \neg R(n))$.

3.1.3 Existential Quantifier

The other quantifier which turns open sentences into statements is the **existential quantifier**. Applying the existential quantifier to the open sentence "x is a real number" produces the statement "There is an x such that x is a real number," or perhaps more succinctly, "Real numbers exist." In mathematical notation:

$$(\exists x)P(x)$$

Again, we have an issue with scope which we can avoid by writing something like the following:

$$(\exists x \in \mathbb{N})P(x)$$

Exercise 3.1.4. Let $P(n)$ be the sentence "n is odd," $Q(n)$ be the sentence "n is greater than 4," and $R(n)$ be the sentence "n is negative." Translate the following symbols into elegant English sentences.

(a) $(\exists n \in \mathbb{N})P(n)$.

(b) $(\exists n \in \mathbb{Z})(\neg Q(n))$.

(c) $(\exists n \in \mathbb{N})(R(n) \vee P(n))$.

(d) $(\exists n \in \mathbb{Z})(Q(n) \wedge \neg R(n))$.

Exercise 3.1.5. Let A and B be sets. Write the statement $A \subset B$ using quantifiers.

Exercise 3.1.6. Let A and B be sets. Write the statement $A \cap B \neq \varnothing$ using quantifiers.

3.1.4 Negations of Quantifiers

For concreteness, let A be a set and let $P(x)$ be an open sentence. Now consider the statement $(\forall x \in A)P(x)$. What would it take for this statement to be false? Well, there would have to be some $x \in A$ so that for this particular x the statement $P(x)$ is false. This sounds like an existential quantifier. Thus, we have the following:

$$\neg(\forall x)P(x) \iff (\exists x \in A)\neg P(x)$$

Now consider the statement $(\exists x \in A)P(x)$. What would it take for this statement to be false? Well, $P(x)$ would need to be false for every $x \in A$. Thus, we have the following:

$$\neg(\exists x)P(x) \iff (\forall x \in A)\neg P(x)$$

Exercise 3.1.7. Let $P(n)$ be the sentence "n is odd," $Q(n)$ be the sentence "n is greater than 4," and $R(n)$ be the sentence "n is negative." Negate the following sentences. Then write their negations in English.

(a) $(\forall n \in \mathbb{N})P(n)$.

(b) $(\exists n \in \mathbb{Z})(\neg Q(n))$.

(c) $(\forall n \in \mathbb{N})(R(n) \vee P(n))$.

(d) $(\exists n \in \mathbb{Z})(Q(n) \wedge \neg R(n))$

3.2 More on Quantifiers

This worksheet discusses material corresponding roughly to section 4.4 of your textbook.

3.2.1 Review

Recall that we denote open sentences with the notation $P(x)$ or $Q(n)$. Also, we have the universal quantifier $\forall$ and the existential quantifier $\exists$ which we use to turn open sentences into statements. Thus we can have statements like the following:

$$(\forall x \in A)(P(x) \wedge Q(x))$$

$$(\exists x \in A)(P(x) \to Q(x))$$

When negating a quantified statement, we adhere to the following rules:

$$\neg(\forall x \in A)(P(x)) \iff (\exists x \in A)(\neg P(x))$$

$$\neg(\exists x \in A)(P(x)) \iff (\forall x \in A)(\neg P(x))$$

3.2.2 Sentences with multiple free variables

Of course, we can consider open sentences with many free variables. These are sentences like "x is a y" and "A is a subset of B." A more exotic example is "If x says y, then z gets upset." Of course, we denote these open sentences in the same way we would functions of multiple variables. So the sentence "x is a y" could be denoted $P(x, y)$, "A is a subset of B" denoted $Q(A, B)$, and "If x says y, then z gets upset" denoted $R(x, y, z)$.

Exercise 3.2.1. Write the following statements in English. $P(x, y)$, $Q(A, B)$, and $R(x, y, z)$ are the sentences from the preceding paragraph.

(a) $P(\text{Fluffy}, \text{dog})$

(b) $Q(\mathbb{N}, \mathbb{R})$

(c) $R(\text{Sally}, \text{I did it again}, \text{Mary})$.

3.2.3 Quantifiers in such sentences

Of course, the universal and existential quantifiers can be applied to sentences with more than one free variable. The rule of thumb is that to obtain a statement by applying quantifiers, we need one quantifier per free variable.

Exercise 3.2.2. Let $P(x, y)$ be the sentence $x^2 = y$. Write the following out in English. Also, decide if the statement is true or false.

(a) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})P(x, y)$

(b) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})P(x, y)$

(c) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})P(x, y)$

(d) $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})P(x, y)$

3.2.4 Negations of such

Negations work much the same with quantified sentences with multiple free variables as they do in the single variable case, you just work "one quantifier at a time." For example, if we have a sentence $P(x, y, z)$ and the statement $(\forall x \in X)(\forall y \in Y)(\exists z \in Z)P(z, y, x)$, then we negate as follows:

$$\neg(\forall x \in X)(\forall y \in Y)(\exists z \in Z)P(z, y, x)$$
$$(\exists x \in X)\neg(\forall y \in Y)(\exists z \in Z)P(z, y, x)$$
$$(\exists x \in X)(\exists y \in Y)\neg(\exists z \in Z)P(z, y, x)$$
$$(\exists x \in X)(\exists y \in Y)(\forall z \in Z)\neg P(z, y, x)$$

Exercise 3.2.3. Let $P(x, y)$ be the sentence $x^2 = y$. Negate each of the following. Write each statement in English. Also, determine whether each negated statement is true or false.

(a) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})P(x, y)$

(b) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})P(x, y)$

(c) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})P(x, y)$

(d) $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})P(x, y)$

Exercise 3.2.4. Recall from calculus that if f is a real valued function of a single real variable, then $\lim_{x \to a} f(x) = L$ means that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$. This is the definition of the **limit of f as x approaches a**.

(a) Express $\lim_{x \to a} f(x) = L$ using quantifiers. It may be useful to denote the set of positive real numbers as $\mathbb{R}^+$.

(b) Express $\lim_{x \to a} f(x) \neq L$ using quantifiers.

It's often convenient to abbreviate $(\forall x \in X)(\forall y \in X)$ as $(\forall x, y \in X)$ and similarly abbreviate $(\exists x \in X)(\exists y \in X)$ as $(\exists x, y \in X)$.

Exercise 3.2.5. Negate each of the following statements below.

(a) $(\forall P, Q \in X)(P \neq Q \to (\exists \ell \in L \text{ such that } P \in \ell \text{ and } Q \in \ell.))$

(b) $(\forall \ell \in L)(\exists P, Q \in X)(P \neq Q \text{ and } P \in \ell \text{ and } Q \in \ell.)$

(c) $(\forall \ell \in L)(\forall P \notin \ell)(\exists m \in L)(P \in m \text{ and } \ell \cap m = \varnothing)$.

Chapter 4

Relations and Functions

4.1 Cartesian Product of Sets

This worksheet discusses material corresponding roughly to section 5.1 of your textbook.

An **ordered pair** of elements is a list of two elements, denoted (a, b). Two ordered pairs (a, b) and (c, d) are **equal** if $a = c$ and $b = d$. That is, $(a, b) = (c, d)$ if and only if both $a = c$ and $b = d$.

Exercise 4.1.1. Determine whether the statements below are true or false.

(a) $(1, 2) = (2, 1)$

(b) $(\varnothing, \mathbb{N}) = (\mathbb{N}, \varnothing)$

(c) $(\{1, 2, 3, 4\}, \{1, 2\}) = (\{2, 1, 4, 3\}, \{2, 1\})$

(d) $(\mathbb{R}, \mathbb{R}) = (\mathbb{R}, \mathbb{R})$

(e) $(1, 0) = (\sin(\pi/2), \cos(\pi/2))$

(f) $1 = (1, 0)$

(g) $1 = (0, 1)$

(h) $(x, y) = ((x, y), 0)$

With this notion of ordered pairs in mind, we make the following definition.

Definition 4.1.2. Let A and B be sets. Then the **Cartesian product** of A with B is the set $A \times B$ where

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

That is, the Cartesian product of A with B is the set of all ordered pairs where the first entry comes from A and the second entry comes from B.

Exercise 4.1.3. Compute $A \times B$ for the sets A and B below.

(a) $A = \{1, 2\}$, $B = \{x, y\}$.

(b) $B = \{1, 2\}$, $A = \{x, y\}$.

(c) $A = \{-1, 0, 1\}$, $B = \{10\}$.

(d) $A = \mathbb{R}$, $B = \varnothing$.

These examples are giving me some ideas for theorems. Prove these:

Exercise 4.1.4. Let A be a set. Prove that $A \times \varnothing = \varnothing$.

Exercise 4.1.5. Let A, B, and C be sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Exercise 4.1.6. Let A, B, and C be sets. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Exercise 4.1.7. Prove or disprove: $A \times (B - C) = (A \times B) - (A \times C)$.

Exercise 4.1.8. Prove or disprove: $A - (B \times C) = (A - B) \times (A - C)$.

Exercise 4.1.9. Prove or disprove: $A \cup (B \times C) = (A \cup B) \times (A \cup C)$

Exercise 4.1.10. Prove or disprove: $A \cap (B \times C) = (A \cap B) \times (A \cap C)$.

4.2 Divisibility and Relations

This worksheet discusses material corresponding roughly to section 5.1-5.2 of your textbook.

4.2.1 Divisibility

Definition 4.2.1. Let a and b be integers. Then we say a **divides** b, and write $a|b$, if there an integer c such that $ac = b$. That is

$$a|b \Leftrightarrow (\exists c \in \mathbb{Z})(ac = b)$$

Exercise 4.2.2. Prove or disprove.

(a) $3|102$

(b) $5|302$

(c) $-4|-8$

(d) $0|10$

(e) $11|0$

Let a, b, and c be integers.

Exercise 4.2.3. Prove or disprove: Every integer divides zero.

Exercise 4.2.4. Prove or disprove: If $a|b$ and $b|c$, then $a|c$.

Exercise 4.2.5. Prove or disprove: If $a|b$, then $b|a$.

Exercise 4.2.6. Prove or disprove: $a|a$.

4.2.2 Relations

Definition 4.2.7. A **relation** between sets A and B is a subset of $A \times B$.

For example, the notion of divides from above determines a relation between the sets $A = \mathbb{Z}$ and $B = \mathbb{Z}$. So, we can think of '$|$' as a subset of $\mathbb{Z} \times \mathbb{Z}$. Which subset? This one:

$$| \ = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a|b\}$$

For example, the ordered pair of integers $(2, 4)$ is a member of the set $|$ since $2|4$ is true, while $(2, 3)$ is not a member of $|$ since $2|3$ is false.

Exercise 4.2.8. Determine which of these pairs of integers are members of the set $|$.

(a) $(3, 102)$

(b) $(5, 302)$

(c) $(-4, -8)$

(d) $(0, 10)$

(e) $(11, 0)$

Exercise 4.2.9. Prove or disprove: If $(a, b) \in |$, then $(b, a) \in |$.

Exercise 4.2.10. Prove or disprove: $(a, a) \in |$ for every integer a.

Exercise 4.2.11. Prove or disprove: If $(a, b) \in |$ and $(b, c) \in |$, then $(a, c) \in |$.

A perhaps more familiar relation is that of $<$. We can think of $<$ as giving us a relation on the real numbers. That is, $<$ is a subset of $\mathbb{R} \times \mathbb{R}$. Which subset?

$$< = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x < y\}$$

For example, $(\pi, \sqrt{19}) \in <$ since $\pi < \sqrt{19}$ is true. However, $(-1, -\pi)$ is not in $<$ since $-1 < -\pi$ is false.

Exercise 4.2.12. Determine which of these pairs of real numbers are members of the set $<$.

(a) $(3, 102)$

(b) $(5, 302)$

(c) $(-4, -8)$

(d) $(0, 10)$

(e) $(11, 0)$

Exercise 4.2.13. Prove or disprove: If $(a, b) \in <$, then $(b, a) \in <$.

Exercise 4.2.14. Prove or disprove: $(a, a) \in <$ for every real number a.

Exercise 4.2.15. Prove or disprove: If $(a, b) \in <$ and $(b, c) \in <$, then $(a, c) \in <$.

Here is another relation on $\mathbb{R} \times \mathbb{R}$.

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 > 0\}$$

In this case, we'll write xRy to indicate that $(x, y) \in R$.

Exercise 4.2.16. Determine which of these pairs of real numbers are members of the set R.

(a) $(3, 102)$

(b) $(5, 302)$

(c) $(-4, -8)$

(d) $(0, 10)$

(e) $(11, 0)$

Exercise 4.2.17. Prove or disprove: If $(a, b) \in R$, then $(b, a) \in R$.

Exercise 4.2.18. Prove or disprove: $(a, a) \in R$ for every real number a.

Exercise 4.2.19. Prove or disprove: If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

4.3 Relations

This worksheet discusses material corresponding roughly to section 5.2-5.3 of your textbook.

4.3.1 Relations

Recall that a **relation** R between sets A and B is a subset $R \subset A \times B$. If the ordered pair $(a, b) \in R$, then we write aRb.

If R is a relation between a set A and itself (i.e. $R \subset A \times A$), then we say R is a **relation on** A.

Exercise 4.3.1. Consider the relation R on $\mathbb{N}$ given by $R = \{(a, b) \mid a + 5 < 2b\}$.

(a) Is $4R3$?

(b) Prove or disprove: $\forall n \in N$, nRn.

(c) Prove or disprove: $\forall n, m \in N$, if nRm, then mRn.

(d) Prove or disprove: $\forall n, m, p \in N$, if nRm and mRp, then nRp.

Exercise 4.3.2. Consider the relation '$\equiv \mod 5$' on $\mathbb{Z}$ given by the following. For integers n and m, then $n \equiv m \mod 5$ if and only if $n - m$ is divisible by 5. You pronounce $n \equiv m \mod 5$ by saying "n is congruent to m modulo 5."

(a) Determine whether or not $4 \equiv -17 \mod 5$.

(b) Determine whether or not $1 \equiv 101 \mod 5$.

(c) Determine all integers n for which $n \equiv 0 \mod 5$.

A relation R on a set A is **reflexive** if aRa for all $a \in A$. That is, $(a, a) \in A$ for all $a \in A$.

Exercise 4.3.3. Prove or disprove: The relation $\equiv \mod 5$ is reflexive.

A relation R on a set A is **symmetric** if for all $a, b \in A$, aRb if and only if bRa.

Exercise 4.3.4. Prove or disprove: The relation $\equiv \mod 5$ is symmetric.

A relation R on a set A is **transitive** if for all $a, b, c \in A$, aRb and bRc implies aRc.

Exercise 4.3.5. Prove or disprove: The relation $\equiv \mod 5$ is transitive.

Exercise 4.3.6. Prove or disprove: The relation $|$ on $\mathbb{Z}$ is reflexive.

Exercise 4.3.7. Prove or disprove: The relation $|$ on $\mathbb{Z}$ is symmetric.

Exercise 4.3.8. Prove or disprove: The relation $|$ on $\mathbb{Z}$ is transitive.

Exercise 4.3.9. Prove or disprove: The relation R from Exercise 4.3.1 is reflexive.

Exercise 4.3.10. Prove or disprove: The relation R from Exercise 4.3.1 is symmetric.

Exercise 4.3.11. Prove or disprove: The relation R from Exercise 4.3.1 is transitive.

4.3.2 Equivalence Relations

A relation R on a set A is an **equivalence relation** if it is reflexive, symmetric, and transitive.

Exercise 4.3.12. Determine which of the relations, if any, presented in this worksheet are equivalence relations.

Exercise 4.3.13. Consider the relation $\equiv \mod 5$ on $\mathbb{Z}$.

(a) Determine the set $\{n \in Z \mid n \equiv 0 \mod 5\}$. Denote this set by $[0]_5$.

(b) Determine the set $\{n \in Z \mid n \equiv 1 \mod 5\}$. Denote this set by $[1]_5$.

(c) Determine the set $\{n \in Z \mid n \equiv 2 \mod 5\}$. Denote this set by $[2]_5$.

(d) Determine the set $\{n \in Z \mid n \equiv 3 \mod 5\}$. Denote this set by $[3]_5$.

(e) Determine the set $\{n \in Z \mid n \equiv 4 \mod 5\}$. Denote this set by $[4]_5$.

(f) Determine the set $\{n \in Z \mid n \equiv 5 \mod 5\}$. Denote this set by $[5]_5$.

(g) Prove or disprove: $[0]_5 = [1]_5$.

(h) Continuing with the above notation, let $[k]_5 = \{n \in \mathbb{Z} \mid n \equiv k \mod 5\}$. Determine when $[k]_5 = [\ell]_5$.

(i) Is $[-1029]_5 = [2047]_5$?

4.4 Modular Arithmetic: An Equivalence Relation

This worksheet discusses material corresponding roughly to section 5.2-5.3 of your textbook.

Recall that from the last worksheet, we had considered the relation $\equiv \mod 5$ on $\mathbb{Z}$ given by $a \equiv b \mod 5$ if and only if $a - b$ is divisible by 5. There is nothing special here about the number 5. We can just as well define a relation on $\mathbb{Z}$ called $\equiv \mod n$ where n is any positive integer.

Definition 4.4.1. If $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$, then $a \equiv b \mod n$ if and only if $a - b$ is divisible by n.

Exercise 4.4.2. Determine whether or not the following are true.

(a) $4 \equiv 203 \mod 17$.

(b) $-6 \equiv 102 \mod 3$.

(c) $10 \equiv 10 \mod 203$.

(d) $-7 \equiv -19 \mod 12$.

Recall that a relation R on a set A is

- **reflexive** if for every $a \in A$ aRa,
- **symmetric** if for every $a, b \in A$, aRb implies bRa,
- **transitive** if for every $a, b, c \in A$, aRb and bRc implies aRc,
- an **equivalence relation** if it is reflexive, symmetric, and transitive.

Exercise 4.4.3. Let $n \in \mathbb{N}$ be a fixed positive integer. Prove the following theorem.

Theorem 4.4.4. The relation $\equiv \mod n$ is an equivalence relation.

Let n still be fixed. Following notation from the last worksheet still, let

$$[k]_n = \{\ell \in \mathbb{Z} \mid \ell \equiv k \mod n\}$$

Exercise 4.4.5. To be concrete, let $n = 3$.

(a) Determine $[0]_3$, $[1]_3$, and $[2]_3$. Write each of these sets both using the definition above, and also as a list of their elements.

(b) The set $[7]_3$ is equal to one of the sets in part (a). Which one?

(c) The set $[-7]_3$ is equal to one of the sets in part (a). Which one?

(d) Prove that $[k]_3 = [\ell]_3$ if and only if $k \equiv \ell \mod 3$.

(e) Prove that the sets $[0]_3$, $[1]_3$, and $[2]_3$ are pairwise disjoint. (This means that if you take any two of them, their intersection is empty.)

(f) Prove that if $k \in [\ell]_3$, then $k + 3 \in [\ell]_3$ and also $k - 3 \in [\ell]_3$.

(g) Prove that if $k \in [\ell]_3$, then $k + 3m \in [\ell]_3$ for any $m \in \mathbb{Z}$.

(h) Convince yourself that for any $k \in \mathbb{Z}$ there is an $m \in \mathbb{Z}$ so that $k + 3m \in \{0, 1, 2\}$. (Proving this essentially amounts to proving what is known as the Division Algorithm. We'll do that later in the course.)

(i) Prove that for each $k \in \mathbb{Z}$, exactly one of the following is true:

- $k \in [0]_3$
- $k \in [1]_3$
- $k \in [2]_3$

(j) Prove that $\mathbb{Z} = [0]_3 \cup [1]_3 \cup [2]_3$.

Ah. So the sets $[0]_3$, $[1]_3$, and $[2]_3$ are pairwise disjoint (from part (e)) and union to give all of $\mathbb{Z}$ (from part (j)). When this happens, we say that theses sets **partition** the integers. Here is the general definition.

Definition 4.4.6. Let A be a set and $\{A_1, A_2, \ldots, A_n\}$ a set of subsets of A. (That is $A_i \subset A$ for each $i \in \{1, \ldots, n\}$.) Then the sets $A_1, \ldots, A_n$ form a **partition** of A if

- $A_i \cap A_j = \varnothing$ if $i \neq j$, and
- $A = \bigcup_{i=1}^{n} A_i$.

The notation in the second bullet point is short hand for "union all of these sets," just like you use Σ notation to indicate "add up all of these numbers." More specifically:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

Exercise 4.4.7. Consider the set $A = \{1, 2, 3, 4, 5\}$. Let $A_1 = \{2, 3, 4\}$, $A_2 = \{1\}$, and $A_3 = \{5\}$. Determine if $\{A_1, A_2, A_3\}$ is a partition of A or not.

Exercise 4.4.8. Let $B = \{a, b, c, d, e, f, g, h\}$. Let $B_1 = \{a, c\}$, $B_2 = \{b, f, g, h\}$, and $B_3 = \{d\}$. Determine if $\{B_1, B_2, B_3\}$ is a partition of B or not.

4.5 Partitions

This worksheet discusses material corresponding roughly to section 5.4 of your textbook.

Recall the following definition.

Definition 4.5.1. Let A be a set and $\{A_1, A_2, \ldots, A_n\}$ a set of subsets of A. (That is $A_i \subset A$ for each $i \in \{1, \ldots, n\}$.) Then the sets $A_1, \ldots, A_n$ form a **partition** of A if

- $A_i \cap A_j = \varnothing$ if $i \neq j$, and
- $A = \bigcup_{i=1}^{n} A_i$.

The notation in the second bullet point is short hand for "union all of these sets," just like you use Σ notation to indicate "add up all of these numbers." More specifically:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

Exercise 4.5.2. Consider the set $A = \{1, 2, 3, 4, 5\}$. Let $A_1 = \{2, 3, 4\}$, $A_2 = \{1\}$, and $A_3 = \{5\}$. Determine if $\{A_1, A_2, A_3\}$ is a partition of A or not.

Exercise 4.5.3. Let $B = \{a, b, c, d, e, f, g, h\}$. Let $B_1 = \{a, c\}$, $B_2 = \{b, f, g, h\}$, and $B_3 = \{d\}$. Determine if $\{B_1, B_2, B_3\}$ is a partition of B or not.

Exercise 4.5.4. Consider the set $C = \{1, 2, 3\}$.

1. Find all partitions of C into two sets.

2. Find all partitions of C into three sets.

3. Find all partitions of C into one set.

4. Find all partitions of C into zero sets.

Exercise 4.5.5. Recall that $\equiv \mod 3$ is an equivalence relation on the integers. Also recall that we have the following notation:

$$[k]_3 = \{n \in \mathbb{Z} \mid n \equiv k \mod 3\}$$

(a) Prove that the sets $[0]_3$, $[1]_3$, and $[2]_3$ partition the integers.

(b) Find integers a_1, a_2, a_3, and a_4 so that $\{[a_1]_4, [a_2]_4, [a_3]_4, [a_4]_4\}$ is a partition of the integers.

Exercise 4.5.6. Suppose $\{A_1, \ldots, A_n\}$ is a partition of the set A. Define a relation R on A by the following:

$$aRb \text{ if and only if } \exists i \in \{1, \ldots, n\} \text{ such that } a \in A_i \text{ and } b \in A_i$$

(a) Prove that R is reflexive.

(b) Prove that R is symmetric.

(c) Prove that R is transitive.

(d) Prove that R is an equivalence relation.

4.6 Partitions and Equivalence Classes

This worksheet discusses material corresponding roughly to section 5.4 of your textbook.

4.6.1 Equivalence Classes

If $\sim$ is an equivalence relation on a set A, then for any $a \in A$ we define **the equivalence class of** a, denoted $[a]$, to be

$$[a] = \{x \in A \mid x \sim a\}$$

That is, $[a]$ is the set of all elements of A which are related to a by the relation $\sim$.

Exercise 4.6.1. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and define a relation $\sim$ on A by $a \sim b$ if and only if $a - b$ is even.

(a) Observe that $\sim$ is an equivalence relation on A.

(b) For each element $a \in A$ determine $[a]$. That is, write out explicitly the set $[a]$ for each of the nine elements of A.

(c) How many distinct equivalence classes are there for this relation $\sim$?

(d) Create the a set called $A/{\sim}$ which is the set of all equivalence classes for the equivalence relation $\sim$ on A. That is, complete the line below:

$$A/{\sim} = \Big\{$$

Recall the following definition.

Definition 4.6.2. Let A be a set and $\{A_1, A_2, \ldots, A_n\}$ a set of subsets of A. (That is $A_i \subset A$ for each $i \in \{1, \ldots, n\}$.) Then the sets $A_1, \ldots, A_n$ form a **partition** of A if

- $A_i \cap A_j = \varnothing$ if $i \neq j$, and
- $A = \bigcup_{i=1}^{n} A_i$.

(e) Determine whether or not $A/\sim$ is a partition of A.

Exercise 4.6.3. Repeat the previous exercise with the set of letters

$$\Omega = \{ \text{A, C, D, E, F, G, H, I, J, K,L, M, N, O, P ,Q, R, S, T, U, V, W, X, Y, Z} \}$$

with the relation $\sim$ given by $\alpha \sim \beta$ if and only if the letters α and β have the same number of "end points." (For example, M has two end points and E has three, so $M \not\sim E$, but $M \sim Q$.)

4.6.2 Equivalence Relations and Partitions

The phenomenon in the previous two exercises exhibits a relationship between an equivalence relation on a set A and a partition of the set A. That is, every equivalence relation determines a partition, and the partition in turn determines an equivalence relation. You'll now work to see the relationship between these two. First, you'll prove the following theorem.

Theorem 4.6.4. If $\sim$ is an equivalence relation on a nonempty set A, then $A/\sim$ is a partition of A.

Exercise 4.6.5. Prove the theorem by taking the following steps.

(a) Show that if $a \in A$, then $a \in [a]$.

(b) Show that $\bigcup_{a \in A} [a] = A$.

(c) Let $a, b, \in A$. Show that if $[a] \cap [b] \neq \varnothing$, then $[a] = [b]$. (Remember, $[a]$ and $[b]$ are *sets*, so this is proving equality of sets.)

(d) Conclude that if $[a] \neq [b]$, then $[a] \cap [b] = \varnothing$.

(e) Conclude that $A/\sim$ is a partition of A.

We can do the reverse as well.

Exercise 4.6.6. Let P be a partition of A. Then we can define a relation $\sim$ on A by $a \sim b$ if and only if there is a set $B \in P$ so that $a \in B$ and $b \in B$.

(a) Show that $\sim$ is reflexive.

(b) Show that $\sim$ is symmetric.

(c) Show that $\sim$ is transitive.

(d) Conclude that $\sim$ is an equivalence relation.

4.7 Functions

This worksheet discusses material corresponding roughly to section 5.7 of your textbook.

4.7.1 Definition

Let A and B be sets. Recall that a relation from A to B is a subset of $A \times B$.

Then a **function** f from A to B is a relation from A to B so that

- $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in f$, and
- if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$.

The first condition states that each element of A is represented as the first element of at least one ordered pair in f. The second condition states that each element of A is represented in no more than one ordered pair in f. So each element of A is represented in *exactly one* ordered pair in f. Instead of writing $(a, b) \in f$, we typically use the familiar notation

$$f(a) = b.$$

Also, when we have a function f from A to B, we will denote this by writing $f : A \to B$. The set A is called the **domain** of f and the set B is called the **codomain** of f.

Exercise 4.7.1. Determine whether or not the relations below are functions. If it is a function, identify the domain and codomian.

(a) $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2\}$

(b) $g = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y^2\}$

(c) $h = \{(n, x) \in \mathbb{N} \times \mathbb{R} \mid nx = 1\}$

(d) $k = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid n|m\}$

4.7.2 Properties of Functions

If $f\colon A \to B$, then the **range** of f is the set $\{b \in B \mid \exists a \in A \text{ such that } f(a) = b\}$.

Exercise 4.7.2. Determine the range of each of the functions in the previous exercise.

A function $f\colon A \to B$ is called **injective** or **one-to-one** if $f(x) = f(y) \to x = y$. Equivalently, f is injective if $x \neq y \to f(x) \neq f(y)$. (So injective functions map different points to different points.)

Exercise 4.7.3. Determine which of the functions below are injective. Also, determine the range of each function.

(a) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

(b) $g\colon [0, \infty) \to \mathbb{R}$, $g(x) = x^2$.

(c) $h\colon \mathbb{N} \to \mathbb{R}$, $h(n) = n^2$.

A function $f\colon A \to B$ is called **surjective** or **onto** if for every $b \in B$ there is an $a \in A$ so that $f(a) = b$. That is, every point in the codomain is the output of f for some input.

Exercise 4.7.4. Determine which of the function below are surjective. Also, determine the range of each function.

(a) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

(b) $g\colon \mathbb{R} \to [0, \infty)$, $g(x) = x^2$.

(c) $h\colon \mathbb{R} \to \mathbb{Z}$, $h(x) = \lfloor x \rfloor$. (This is the floor function, which rounds x down to the nearest integer.)

(d) $k\colon [0, 1] \to [0, 1]$, $k(t) = 1 - t$.

A function $f\colon A \to B$ is a **bijection** if it is both an injection and a surjection.

Exercise 4.7.5. Determine which of the functions in the previous exercise are bijections.

Exercise 4.7.6. Let $A = \{1, 2\}$ and $B = \{\alpha, \beta, \gamma\}$.

(a) Determine all injective functions $f\colon A \to B$.

(b) Are there any surjective functions $f\colon A \to B$? Explain.

4.8 Inverses, Composition, Images, and Inverse Images

This worksheet discusses material corresponding roughly to sections 5.8-5.9 of your textbook.

4.8.1 Inverse Relations

Let $f\colon A \to B$ be a function. Then we define the **inverse relation** f^{-1} to be the relation from B to A given by

$$f^{-1} = \{(b, a) \in B \times A \mid (a, b) \in f\}$$

In general, f^{-1} need not be a function.

Exercise 4.8.1. Let $A = \{1, 2, 3\}$ and $B = \{\alpha, \beta, \gamma\}$. Let $f\colon A \to B$ be the function $f = \{(1, \beta), (2, \gamma), (3, \beta)\}$.

(a) Determine f^{-1}. Recall from the definition above that $f^{-1} \subset B \times A$ and so should be a set of ordered pairs.

(b) Prove or disprove: The function f is injective.

(c) Prove or disprove: The function f is surjective.

(d) Prove or disprove: The relation f^{-1} is a function.

Exercise 4.8.2. Prove: $f\colon A \to B$ is bijective if and only if $f^{-1}\colon B \to A$. (That is, f is bijective if and only if f^{-1} is a function.)

4.8.2 Composition

Let $f\colon A \to B$ and $g\colon B \to C$. Then we can define a new function with domain A and codomain C. This function is called the **composition** of f and g, denoted

$$g \circ f\colon A \to C$$

and is given by $(g \circ f)(a) = g(f(a))$.

Exercise 4.8.3. Let $A = \{1, 2, 3\}$, $B = \{\alpha, \beta, \gamma\}$, and $C = \{w, x, y, z\}$. Let $f\colon A \to B$ be given by $f(1) = \gamma$, $f(2) = \alpha$, and $f(3) = \alpha$. Let $g\colon B \to C$ be given by $g(\alpha) = y$, $g(\beta) = z$, and $g(\gamma) = w$. Determine the function $g \circ f\colon A \to C$.

Given any set A, there is a canonical[1] function from A to A: the **identity function**, denoted $i_A\colon A \to A$. This function is given by $i_A(a) = a$ for all $a \in A$.

Exercise 4.8.4. Prove: If $f\colon A \to B$ is bijective, then $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

[1] In mathematics, the word "canonical" is often used to mean "archetypal" or "standard" or "the one you would think of that applies universally." In the usage here, it means that this construction applies universally for all sets A.

4.8.3 Images and Inverse Images

Let $f\colon A \to B$ be a function. If $C \subset A$, then we define the **image of** C **under** f to be the set $f[C] \subset B$ given by

$$f[C] = \{b \in B \mid \exists c \in C \text{ such that } f(c) = b\}$$

Equivalently,

$$f[C] = \{f(c) \in B \mid c \in C\}$$

If $D \subset B$, then we define the **preimage of** D **under** f to be the set $f^{-1}[D] \subset A$ given by

$$f^{-1}[D] = \{a \in A \mid f(a) \in D\}$$

Exercise 4.8.5. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be $f(n) = n^2 + 1$.

(a) Let $C = \{1, 2, 3\}$. Determine $f[C]$.

(b) Let $D = \{-5, -2, 0, 2, 5\}$. Determine $f[C]$.

(c) Let $E = \{2, 5, 10\}$. Determine $f^{-1}[E]$.

(d) Let $F = \{0, -6, 20\}$. Determine $f^{-1}[F]$.

4.9 Images and Inverse Images

This worksheet discusses material corresponding roughly to sections 5.8-5.9 of your textbook.

Recall that if $f\colon A \to B$ and $C \subset A$, then $f[C] = \{f(c) \in B \mid c \in C\}$. Also, if $D \subset B$, then $f^{-1}[D] = \{a \in A \mid f(a) \in D\}$. The set $f[C]$ is called the **image of** C under f and the set $f^{-1}[D]$ is called the **preimage of** D under f.

Exercise 4.9.1. Let $f\colon \mathbb{N} \to \mathbb{N}$ be the function given by $f(n) = 3n + 2$ and $g\colon \mathbb{N} \to \mathbb{N}$ be the function given by $g(n) = 2n + 1$.

(a) Let $C = \{1, 4, 6, 10\}$. Determine $f[C]$ and $g[C]$.

(b) Let $D = \{5, 11, 17\}$. Determine $f^{-1}[D]$ and $g^{-1}[D]$.

(c) Let $E = \{2, 8, 20\}$. Determine $f[E]$, $g[E]$, $f^{-1}[E]$, and $g^{-1}[E]$.

Exercise 4.9.2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2 + 1$. Let $A = \{0, 1, 2\} \subset \mathbb{R}$.

(a) Determine $f[A]$.

(b) Determine $f^{-1}[f[A]]$.

(c) Determine $f^{-1}[A]$.

(d) Determine $f[f^{-1}[A]]$.

Exercise 4.9.3. Let $f\colon A \to B$ be a function, $C \subset A$, and $D \subset B$.

(a) Prove or disprove: $f^{-1}[f[C]] \subset C$.

(b) Prove or disprove: $C \subset f^{-1}[f[C]]$.

(c) Prove or disprove: $f[f^{-1}[D]] \subset D$.

(d) Prove or disprove: $D \subset f[f^{-1}[D]]$.

Exercise 4.9.4. Revisit each of the statements in the previous exercise under the assumption that

(a) f is injective.

(b) f is surjective.

Chapter 5

Cardinality

5.1 Cardinality

This worksheet discusses material corresponding roughly to section 6.1 of your textbook.

We are about to discuss the sizes of sets.

Definition 5.1.1. Let A and B be sets. We say that A and B have the same **cardinality** if there is a bijection $f\colon A \to B$. In this case, we write $|A| = |B|$. The symbol $|A|$ is read "the cardinality of A."

For now, when discussing the size of sets, we will avoid using terms like "finite" and "infinite" until such terms are defined. We'll also avoid saying something like "the set {1,6, 102, 45678} has cardinality 4" since this really hasn't been defined yet. To be clear, **all explanations of cardinalities of sets below must appeal to the definition above**.

Exercise 5.1.2. Let $A = \{a, b, c\}$ and $B = \{\omega, \eta, \nu\}$. Let $f\colon A \to B$ be the function given by $f(a) = \eta$, $f(b) = \nu$, and $f(c) = \omega$.

(a) Explain why f is a bijection.

(b) Do A and B have the same cardinality? Explain.

(c) Find another set C so that A and C have the same cardinality. Do C and B have the same cardinality? Explain.

(d) Does the set $D = \{\rho, \tau\}$ have the same cardinality as A? That is, is $|D| = |A|$? Explain.

Exercise 5.1.3. Let $X = \{3, 4, 5, 6, \dots\}$ and $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. Let $g\colon \mathbb{N} \to X$ be the function $g(n) = n + 3$.

(a) Prove that g is a bijection.

(b) Prove that $|X| = |\mathbb{N}|$.

Exercise 5.1.4. Let E denote the set of even integers. Let $f\colon \mathbb{Z} \to E$ be the function $f(n) = 2n$.

(a) Prove that f is bijective.

(b) Prove that $|\mathbb{Z}| = |E|$.

Exercise 5.1.5. Consider the sets $\mathbb{N}$ and $\mathbb{Z}$. In the left column below are the elements of $\mathbb{N}$, and in the right column are the elements of $\mathbb{Z}$.

(a) Create a function $f\colon \mathbb{N} \to \mathbb{Z}$ which is a bijection. Indicate the function by drawing arrows. (For example, if you want $f(4) = 3$, then draw an arrow from 4 on the left to 3 on the right.)

	⋮
	-3
	-2
	-1
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
⋮	⋮

(b) Write an explicit formula for your function.

(c) Conclude that $|\mathbb{N}| = |\mathbb{Z}|$.

Exercise 5.1.6. Let A, B, and C be sets.

(a) Prove that $|A| = |A|$.

(b) Prove that if $|A| = |B|$, then $|B| = |A|$.

(c) Prove that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Exercise 5.1.7. Let A be the set of negative integers. Prove that $|A| = |\mathbb{N}|$.

Exercise 5.1.8. Let O be the set of odd integers. Prove that $|O| = |\mathbb{Z}|$.

5.2 ℕ

Today is a good day for you: you are about to construct the natural numbers, ℕ. To do so, you must abandon all notions of numbers.

Exercise 5.2.1. Forget, temporarily, all notions you have of integers, rational numbers, real numbers, complex numbers, or any other sort of numbers you may be familiar with. Take a brief moment of silence to do so.

If at any point today you notice yourself speaking of numbers, return to this exercise.

If at any point today you notice a classmate speaking of numbers, have that classmate return to this exercise.

We need only these definitions.*

Definition. The set $\varnothing$ is the **empty set**. The empty set is uniquely characterized by the fact that the statement "$a \in \varnothing$" is false for all a.

Definition. Let A be a set. Then the **successor** of A, denoted $S(A)$, is the set $S(A) = A \cup \{A\}$.

Exercise. Compute $S(\varnothing)$.

Exercise. Compute $S(S(\varnothing))$.

Exercise. Compute $S(S(S(\varnothing)))$.

*Actually, we need a little more. But to say more would take us too far afield.

Definition. We make the following definitions.

- $0 = \varnothing$.
- $1 = S(0) = \{\varnothing\} = \{0\}$.
- $2 = S(1) = \{\varnothing, \{\varnothing\}\} = \{0, 1\}$.
- $3 = S(2) = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = \{0, 1, 2\}$.
- $4 = S(3) = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\} = \{0, 1, 2, 3\}$.
- $5 = S(4) = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\}\} = \{0, 1, 2, 3, 4\}$.
- etc.

It is important to keep in mind that the symbols 0, 1, 2, etc., refer to **sets**.

Exercise. Show that all of the following are true.

- $0 \in 3$.
- $0 \subset 3$.
- $1 \in 3$
- $1 \subset 3$.
- $2 \in 3$.
- $2 \subset 3$.

Definition. The **set of natural numbers** is the set $\mathbb{N}$ such that the following are true:

- $0 \in \mathbb{N}$.
- If $n \in \mathbb{N}$, then $S(n) \in \mathbb{N}$.
- If $n \in \mathbb{N}$, then either $n = 0$, or n is a successor of 0.

The first of these conditions states that $\mathbb{N}$ is non-empty. The second states that all successors of all successors of 0 are in $\mathbb{N}$. The third states that these are the only members of $\mathbb{N}$.

So, each natural number is a set of sets, and the set of all natural numbers is a set of sets of sets (of sets of sets of...).

We now regain some familiar structures of the natural numbers.

Definition. Let $n, m \in \mathbb{N}$. Then we say n **is less than or equal to** m if $n \subseteq m$. In this case we write $n \leq m$, and we also say that m **is greater than or equal to** n and write $m \geq n$.

Exercise. Prove: $2 \leq 4$.

Exercise. Prove: Let $n, m \in \mathbb{N}$. If $n \leq m$ and $m \leq n$, then $n = m$.

Let A and B be sets. Recall the following definitions.

Definition. A **function** $f\colon A \to B$ is a subset $f \subset A \times B$ such that

- $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in f$, and
- if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$.

Definition. A function $f\colon A \to B$ is **injective** if $f(x) = f(y)$ implies $x = y$.

Definition. A function $f\colon A \to B$ is **surjective** if for every $b \in B$ there is an $a \in A$ so that $f(a) = b$.

Definition. A function $f\colon A \to B$ is **bijective** if it is both injective and surjective.

Definition. Let A and B be sets. Then we say that A **has the same cardinality as** B if there is a bijective function $f\colon A \to B$.

Now, we introduce some new ways to describe the relation of the cardinalities of one set to another.

Definition. If there is an injection $f\colon A \to B$, then we say **the cardinality of** A **is less than or equal to the cardinality of** B and write $|A| \leq |B|$.

Definition. If there is a surjection $f\colon A \to B$, then we say **the cardinality of** A **is greater than or equal to the cardinality of** B and write $|A| \geq |B|$.

Exercise. Prove that $|1| \leq |2|$. Also prove that $|2| \geq |1|$.

Definition. If $A \subset B$, then there is a canonical function $i_{A,B}\colon A \to B$ given by $i_{A,B}(a) = a$ for all $a \in A$. This function $i_{A,B}$ is called the **inclusion** of A into B.

Exercise. Prove that if $A \subset B$, then $i_{A,B}$ is an injection.

Exercise. Let $n, m \in \mathbb{N}$. Prove that if $n \leq m$ then $|n| \leq |m|$.

5.3 Finite Sets and Total Order

Recall the following definitions.

Definition 5.3.1. Let A be a set. Then the **successor** of A is the set $S(A) = A \cup \{A\}$.

- $0 = \varnothing$.
- $1 = S(0) = \{\varnothing\} = \{0\}$.
- $2 = S(1) = \{\varnothing, \{\varnothing\}\} = \{0, 1\}$.
- $3 = S(2) = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = \{0, 1, 2\}$.
- $4 = S(3) = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\} = \{0, 1, 2, 3\}$.
- $5 = S(4) = \{0, 1, 2, 3, 4\}$.
- etc.

Definition 5.3.2. The **set of natural numbers** is the set $\mathbb{N}$ such that the following are true:

- $0 \in \mathbb{N}$.
- If $n \in \mathbb{N}$, then $S(n) \in \mathbb{N}$.
- If $n \in \mathbb{N}$, then either $n = 0$, or n is a successor of 0.

Now, we make a new definition.

Definition 5.3.3. Let A be a set. If there is a bijection $f\colon A \to n$ for some $n \in \mathbb{N}$, then we say that A **has cardinality** n and write $|A| = n$. In this case, we say that A is a **finite** set.

Exercise 5.3.4. Prove that the following sets are finite.

(a) $A = \{a, b, c, d, e\}$

(b) $B = \{\text{apples}, \text{World Peace}, \text{telephone}\}$

(c) $C = \{\$, \%, \P, \S, \copyright, \dagger, \ddagger, \pounds, \star, \bigstar, \&, \#, \bigtriangledown, \bigtriangleup, \cup, \cap, \circ, \triangleleft, \triangleright\}$

Recall the following definitions.

Definition 5.3.5. Let A and B be sets. Then we say that A **has the same cardinality as** B if there is a bijective function $f\colon A \to B$.

Definition 5.3.6. If there is an injection $f\colon A \to B$, then we say **the cardinality of** A **is less than or equal to the cardinality of** B and write $|A| \leq |B|$.

Definition 5.3.7. If there is a surjection $f\colon A \to B$, then we say **the cardinality of** A **is greater than or equal to the cardinality of** B and write $|A| \geq |B|$.

Exercise 5.3.8. Prove that if A and B are finite sets and $n, m \in \mathbb{N}$ with $|A| = n$, $|B| = m$, and $|n| \leq |m|$, then $|A| \leq |B|$.

Exercise 5.3.9. Prove that if A and B are finite sets and $n, m \in \mathbb{N}$ with $|A| = n$, $|B| = m$, and $|n| \geq |m|$, then $|A| \geq |B|$.

Definition 5.3.10. If $A \subset B$, then there is a canonical function $i_{A,B} \colon A \to B$ given by $i_{A,B}(a) = a$ for all $a \in A$. This function $i_{A,B}$ is called the **inclusion** of A into B.

Exercise 5.3.11. Prove that if $A \subset B$, then $i_{A,B}$ is an injection.

Exercise 5.3.12. Suppose that B is finite and $A \subset B$. Prove that A is finite.

Recall this definition of $\leq$ on $\mathbb{N}$.

Definition 5.3.13. Let $n, m \in \mathbb{N}$. If $n \subseteq m$, then we write $n \leq m$ and say n **is less than or equal to** m.

Now, consider this definition.

Definition 5.3.14. Let X be a set. Then a relation $\preceq$ on X is called a **total order** on X if the following conditions hold:

- If $a \preceq b$ and $b \preceq a$, then $a = b$ (antisymmetry).
- If $a \preceq b$ and $b \preceq c$, then $a \preceq c$ (transitivity).
- For all $a, b \in X$, either $a \preceq b$ or $b \preceq a$ (totality).

(The symbol $\preceq$ is pronounced "precedes".)

Exercise 5.3.15. Prove that $\leq$ is a total order on $\mathbb{N}$.

Exercise 5.3.16. Prove that if A is a finite set and there exist $n, m \in \mathbb{N}$ so that $|A| = n$ and $|A| = m$, then $n = m$.

In fact, more is true. Consider the following theorem:

Theorem 5.3.17 (Well-ordering Principal)**.** Let X be a non-empty subset of $\mathbb{N}$. Then X has a least element. That is, there exists an element $x \in X$ so that for every $y \in X$, $x \leq y$.

5.4 Addition in $\mathbb{N}$

We now define the addition operation $+$ on $\mathbb{N}$. We can view $+$ as a function $+\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and as is customary we will write $n + m = k$ in place of the function notation $+(n, m) = k$, and also in place of the relational notation $((n, m), k) \in +$. We require $+$ to have the following properties for all $a, b \in \mathbb{N}$:

- $a + 0 = a$
- $a + S(b) = S(a + b)$

Exercise 5.4.1. Let $a \in \mathbb{N}$.

(a) Prove that $a + 1 = S(a)$.

(b) Prove that $a + 2 = S(S(a))$.

(c) Prove that $a + 3 = S(S(S(a)))$.

Exercise 5.4.2. (a) Prove that $1 + 1 = 2$.

(b) Prove that $2 + 1 = 1 + 2 = 3$.

(c) Prove that $3 + 1 = 1 + 3 = 4$.

(d) Sketch a proof that $a + 1 = 1 + a$ for all $a \in \mathbb{N}$. (Note: To do this properly, we need mathematical induction, which we will have in a week or two.)

Exercise 5.4.3. Let $a, b, c \in \mathbb{N}$.

(a) Prove that $(a + b) + 0 = a + (b + 0) = a + b$.

(b) Prove that $(a + b) + 1 = a + (b + 1)$.

(c) Prove that $(a + b) + 2 = a + (b + 2)$.

(d) Sketch a proof that $(a + b) + c = a + (b + c)$. (Hence, addition is **associative**.) (Again, you need mathematical induction to do this properly.)

Exercise 5.4.4. Let $a, b \in \mathbb{N}$.

(a) Prove that $a + 2 = 2 + a$. (Hint: Use associativity, the fact that $2 = 1 + 1$, and a previous exercise.)

(b) Prove that $a + 3 = 3 + a$. (Hint: Use a variation of the previous hint.)

(c) Sketch a proof that $a+b = b+a$. (Hence, addition is **commutative**.) (Insert comment about induction here.)

Exercise 5.4.5. Prove that if $a + b = 0$, then $a = 0$ and $b = 0$.

Exercise 5.4.6. Let $a, b, c \in \mathbb{N}$. Sketch a proof that if $b + a = c + a$, then $b = c$. (Hence, addition is **cancellative**.)

5.5 Multiplication in $\mathbb{N}$

We now define the multiplication operation $\times$ on $\mathbb{N}$ in a similar manner. Again, we view $\times$ as a function $\times\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, but follow the convention of writing $a \times b$ rather than $\times(a, b)$. We require $\times$ to have the following properties for all $a, b \in \mathbb{N}$:

- $a \times 0 = 0$
- $a \times S(b) = (a \times b) + a$.

Exercise 5.5.1. Let $a, b \in \mathbb{N}$.

(a) Prove that $a \times 1 = a$.

(b) Prove that $a \times 2 = a + a$.

(c) Prove that $a \times 3 = a + a + a$.

(d) Sketch a proof that $a \times b = \overbrace{a + a + \cdots + a}^{b}$

Exercise 5.5.2. Let $a \in \mathbb{N}$. Prove that $a \times 1 = 1 \times a$ by doing the following. (Hence, 1 is the **multiplicative identity**.)

(a) Prove that $a \times 1 = a$. (This should be easy.)

(b) Prove that $1 \times 0 = 0 \times 1$.

(c) Assume that $a = S(b)$ and that you know that $b \times 1 = 1 \times b$. Prove that $a \times 1 = 1 \times a$. (Hint: Write $1 \times a$ as $1 \times S(b)$.)

(d) Observe that you have proven that if $b \times 1 = 1 \times b$, then $S(b) \times 1 = 1 \times S(b)$, and also that $0 \times 1 = 1 \times 0$. The magic words now are "mathematical induction."

Exercise 5.5.3. Let $a \in \mathbb{N}$. Mimic the previous proof to prove that $a \times 0 = 0 \times a$.

Exercise 5.5.4. Let $a, b, c \in \mathbb{N}$. Sketch a proof that $a \times (b + c) = (a \times b) + (a \times c)$ by taking the following steps. (Hence, addition and multiplication satisfy the **distributive law on the left**.)

(a) Prove that $a \times (b + 0) = a \times b + a \times 0$.

(b) Prove that $a \times (b + 1) = a \times b + a \times 1$.

(c) Prove that $a \times (b + 2) = a \times b + a \times 2$. (Hint: Recall that $2 = 1 + 1$ and that addition is associative.)

(d) Assume that $a \times (b + c) = a \times b + a \times c$. Prove that $a \times (b + S(c)) = a \times b + a \times S(c)$.

(e) ...blah blah induction blah blah...

Exercise 5.5.5. Let $a, b, c \in \mathbb{N}$. Sketch a proof that $(a+b) \times c = a \times c + b \times c$ by mimicking the previous proof. (Hence, addition and multiplication satisfy the **distributive law on the right**.)

Exercise 5.5.6. Let $a, b, c \in \mathbb{N}$. Sketch a proof that $(a \times b) \times c = a \times (b \times c)$ by taking the following steps. (Hence, multiplication is **associative**.)

(a) Prove that $(a \times b) \times 0 = a \times (b \times 0) = 0$.

(b) Prove that $(a \times b) \times 1 = a \times (b \times 1)$.

(c) Prove that $(a \times b) \times 2 = a \times (b \times 2)$. (Hint: Recall that $2 = 1 + 1$, that addition is associative, and that multiplication distributes over addition.)

(d) Assume that $(a \times b) \times c = a \times (b \times c)$. Prove that $(a \times b) \times S(c) = a \times (b \times S(c))$.

(e) HOORAY FOR MATHEMATICAL INDUCTION!!!

Exercise 5.5.7. Let $a, b \in \mathbb{N}$. Sketch a proof that $a \times b = b \times a$ by doing the following. (Hence, multiplication is **commutative**.)

(a) Check that you have already proven that both $a \times 0 = 0 \times a$ and $a \times 1 = 1 \times a$.

(b) Prove that $a \times 2 = 2 \times a$. (Hint: Write $2 = 1 + 1$ and use distributivity.)

(c) Prove that $a \times 3 = 3 \times a$. (Hint: Write $3 = 2 + 1$ and use distributivity.)

(d) Prove that $a \times b = b \times a$. (Give me an "I"! Give me an "N"! ...)

5.6 $\mathbb{Z}$

Recall that addition $+$ in $\mathbb{N}$ is defined so that

- $a + 0 = a$ for all $a \in \mathbb{N}$, and
- $a + S(b) = S(a + b)$ for all $a, b \in \mathbb{N}$.

Also recall that you proved that $+$ is associative, commutative, and cancellative, and that if $a + b = 0$, then $a = 0$ and $b = 0$. This means that $\mathbb{N}$ equipped with the addition $+$ lacks **additive inverses**.[1] We want to extend the natural numbers to include additive inverses. Here is what we do. We define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ as follows:

$$(a, b) \sim (c, d) \iff a + d = b + c$$

Exercise 5.6.1. Prove that $\sim$ defined above is an equivalence relation.

[1] In fact, $\mathbb{N}$ equipped with $+$ is an example of something called a **commutative monoid**. This sort of algebraic structure appears quite often in mathematics, and commutative monoids are used in many fields of research mathematics.

We define the set of **integers** $\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/ \sim$. That is, the set of integers is the set of equivalence classes of pairs of natural numbers under the relation $\sim$ above. The elements of $\mathbb{Z}$ are denoted $[(a, b)]$, the equivalence class of $(a, b) \in \mathbb{N} \times \mathbb{N}$.

Exercise 5.6.2. Find three different elements of $\mathbb{N} \times \mathbb{N}$ which are in the same equivalence class as $(2, 7)$. Prove that indeed these elements are in the same equivalence class.

Now, we define addition $+$ on $\mathbb{Z}$ as follows:

$$[(a, b)] + [(c, d)] = [(a + b, c + d)]$$

Exercise 5.6.3. Prove that $+$ on $\mathbb{Z}$ is well defined. That is, prove that if $(a, b) \sim (x, y)$ and $(c, d) \sim (u, v)$, then $(a + c, b + d) \sim (x + u, y + v)$.

We can also define multiplication $\times$ on $\mathbb{Z}$ as follows:

$$[(a, b)] \times [(c, d)] = [(ac + bd, ad + bc)]$$

Exercise 5.6.4. Prove that $\times$ on $\mathbb{Z}$ is well defined. That is, prove that if $(a, b) \sim (x, y)$ and $(c, d) \sim (u, v)$, then $(ac + bd, ad + bc) \sim (xu + yv, xv + yu)$.

Exercise 5.6.5. Prove that if $[(a, b)] \in \mathbb{Z}$, then there is a unique natural number n so that either $[(a, b)] = [(n, 0)]$ or $[(a, b)] = [(0, n)]$.

Exercise 5.6.6. Prove that $[(n, 0)] + [(0, n)] = [(0, 0)]$.

Exercise 5.6.7. Define a function $i\colon \mathbb{N} \to \mathbb{Z}$ by $i(n) = [(n, 0)]$.

(a) Prove that i is an injection.

(b) Prove that $i(n + m) = i(n) + i(m)$. (Notice here that the $+$'s in this equation are referring to two different addition operations. The former is addition in $\mathbb{N}$ and the latter is addition in $\mathbb{Z}$.)

This last exercise shows that $\mathbb{N}$ is embedded in $\mathbb{Z}$ is such a way that addition is preserved. For this reason, we adopt the following notation in $\mathbb{Z}$:

- $n = [(n, 0)] \in \mathbb{Z}$ for $n \in \mathbb{N}$.
- $-n = [(0, n)] \in \mathbb{Z}$ for $n \in \mathbb{N}$.

With this notation, we can define subtraction $-$ in $\mathbb{Z}$ as follows:

$$a - b = a + (-b)$$

Exercise 5.6.8. Prove that $a - a = 0$ and $(a - b) + b = a$ for all $a, b \in \mathbb{Z}$.

5.7 $\mathbb{Q}$

We'll define the rational numbers $\mathbb{Q}$ in a way which is similar to the way we defined the integers $\mathbb{Z}$. We'll consider a relation on ordered pairs of integers, and the rational numbers will be equivalence classes induced by the relation.

To begin, let $\sim$ be the relation on $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ given by

$$(p, q) \sim (r, s) \iff ps = qr$$

Exercise 5.7.1. Prove that $\sim$ is an equivalence relation.

We define the set of **rational numbers** $\mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} - \{0\}))/\sim$. That is, the set of rational numbers is the set of equivalence classes of ordered pairs of integers under the relation $\sim$ above. The elements of $\mathbb{Q}$ are denoted $[(p, q)]$, the equivalence class of $(p, q) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$.

Now, we define addition $+$ on $\mathbb{Q}$ as follows:

$$[(p, q)] + [(r, s)] = [(ps + qr, qs)]$$

Exercise 5.7.2. (a) Show that $(4, 3) \sim (-8, -6)$ and $(-6, 10) \sim (3, -5)$.

(b) Show that $[(4, 3)] + [(-6, 10)] = [(8, 6)] + [(3, -5)]$.

(c) Show more generally that addition is well defined. That is, if $(p, q) \sim (r, s)$ and $(t, u) \sim (v, w)$, then $(p, q) + (t, u) \sim (r, s) + (u, v)$. (You may wish to skip this for now and return to it later.)

We can also define multiplication $\times$ on $\mathbb{Q}$ as follows:

$$[(p,q)] \times [(r,s)] = [(pr,qs)]$$

Exercise 5.7.3. (a) Show that $[(4,3)] \times [(-6,10)] = [(8,6)] \times [(3,-5)]$.

(b) Show more generally that multiplication is well defined. That is, if $(p,q) \sim (r,s)$ and $(t,u) \sim (v,w)$, then $(p,q) \times (t,u) \sim (r,s) \times (u,v)$. (You may wish to skip this for now and return to it later.)

Exercise 5.7.4. Let $i\colon \mathbb{Z} \to \mathbb{Q}$ be given by $i(n) = [(n,1)]$.

(a) Show that i is injective.

(b) Show that $i(a+b) = i(a) + i(b)$ and also $i(a \times b) = i(a) \times i(b)$. (So in a certain sense, this function i preserves addition and multiplication.)

Let's recap. We have the natural numbers $\mathbb{N}$ which we defined as follows:

- $0 = \varnothing$
- $1 = \{0\} = S(0)$
- $2 = \{0, 1\} = S(1) = S(S(0))$
- etc.

Then we defined the integers $\mathbb{Z}$ to be $(\mathbb{N} \times \mathbb{N})/\sim$ where $\sim$ is the equivalence relation $(a, b) \sim (c, d)$ if and only if $a + d = b + c$.

Finally, we define the rational numbers $\mathbb{Q}$ to be $(\mathbb{Z} \times (\mathbb{Z} - \{0\}))/\sim$ where $\sim$ is the equivalence relation $(p, q) \sim (r, s)$ if and only if $ps = qr$.

Hence, rational numbers are equivalence classes of ordered pairs of equivalence classes of ordered pairs of natural numbers.

Exercise 5.7.5. Let $2, 3, 4, 5 \in \mathbb{N}$.

(a) Write out in set notation the natural numbers 2, 3, 4, and 5.

(b) Consider $p = [(3, 4)] \in \mathbb{Z}$. Write $[(3, 4)]$ by replacing 3 and 4 with their corresponding sets. Do the same for $q = [(5, 2)] \in \mathbb{Z}$.

(c) With p and q as above, write out $[(p, q)] \in \mathbb{Q}$ in it's full glory.

(d) Determine the more familiar notation for the rational number $[(p, q)]$ above.

Chapter 6

Induction

6.1 Proof by Induction

The technique of proof by induction works like this: You have a collection of statements $P(n)$, one for each $n \in \mathbb{N}$. If you can show these two things:

- $P(1)$ is true, and
- If $P(k)$ is true, then $P(k+1)$ is true,

then by mathematical induction, $P(n)$ is true for all $n \geq 1$.

Here is a standard example:

Exercise 6.1.1. Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by induction as follows.

(a) Prove that the statement is true when $n = 1$.

(b) Now, assume that the statement is true when $n = k$. Then use that to prove that the statement is true when $n = k + 1$.

Exercise 6.1.2. Prove that the sum of the first n odd natural numbers is n^2.

Exercise 6.1.3. Prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Let A be a set. The **power set** of A is the set $\mathcal{P}(A)$ of all subsets of A.

Exercise 6.1.4. Compute $\mathcal{P}(\varnothing)$, $\mathcal{P}(\{1\})$, $\mathcal{P}(\{1,2\})$, $\mathcal{P}(\{1,2,3\})$ and $\mathcal{P}(\{1,2,3,4\})$.

Exercise 6.1.5. Prove that if A is a finite set with cardinality n, then $\mathcal{P}(A)$ is also a finite set with cardinality 2^n.

Exercise 6.1.6. Let n be a fixed positive natural number. Let $x, y \in \mathbb{Z}$ be integers so that $x \equiv y \mod n$. Prove that $x^k \equiv y^k \mod n$ for every $k \in \mathbb{N}$.

6.2 Infinite Sets

Recall that a set A is **finite** if there is a bijection $f\colon A \to n$ for some $n \in \mathbb{N}$, and in this case we write $|A| = n$. If no such bijection exists for any $n \in \mathbb{N}$, then A is **infinite**.

Before we dig in to infinite sets, recall the following construction. If A is a set, then the **power set** of A is the set $\mathcal{P}(A)$ of all subsets of A.

And, remember how the cardinality of a finite set compares to the cardinality of its power set.

Exercise 6.2.1. Prove that if $|A| = n$, then $|\mathcal{P}(A)| = 2^n$ as follows.

(a) Prove the statement is true for $A = \varnothing$. (This is the base case for induction.)

(b) Assume the statement is true for all sets A with $|A| = k$. (This is the beginning of the inductive step.)

(c) Let A be a set with $|A| = k + 1$. Choose any element $a \in A$ and determine the cardinality of the set $|A - \{a\}|$.

(d) Determine the cardinality of the set $\mathcal{P}(A - \{a\})$.

(e) Show that the two sets $Q = \{X \in \mathcal{P}(A) \mid a \in X\}$ and $R = \{X \in \mathcal{P}(A) \mid a \notin X\}$ form a partition of $\mathcal{P}(A)$.

(f) Show that $|Q| = |\mathcal{P}(A - \{a\})|$.

(g) Show that $|Q| = |R|$.

(h) Determine $|\mathcal{P}(A)|$.

So, the power set of a finite set is larger than that of the original set. (A lot bigger!) But when you have an infinite set A, the power set will also be an infinite set. But bigger?

Recall that if A and B are sets then we write $|A| < |B|$ to mean that there is an injection from A to B, but no surjection from A to B. So, for example, if $A = \{0, 1, 2\}$ and $B = \mathbb{N}$, then $|A| < |\mathbb{N}|$. And, by the exercise above, if A is finite, then $|A| < |\mathcal{P}(A)|$. So, check this out:

Theorem 6.2.2. If A is any set, then $|A| < |\mathcal{P}(A)|$.

Now, you will prove this theorem.

Exercise 6.2.3. Prove the above theorem by doing the following.

(a) Construct an injection $f\colon A \to \mathcal{P}(A)$. (This shows that $|A| \leq |\mathcal{P}(A)|$.)

(b) Now, by way of contradiction, suppose $g\colon A \to \mathcal{P}(A)$ is a surjection. Observe that for any $a \in A$, either $a \in g(a)$ or $a \notin g(a)$.

(c) Let $M = \{a \in A \mid a \notin g(a)\}$. Show that $M \in \mathcal{P}(A)$.

(d) Prove that there is an $x \in A$ so that $g(x) = M$.

(e) Prove that $x \in M$ is a contradiction.

(f) Prove that $x \notin M$ is a contradiction.

(g) Conclude that no such function g exists, and so $|A| < |\mathcal{P}(A)|$.

Exercise 6.2.4. Prove that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$.

Wait. The natural numbers $\mathbb{N}$ is an infinite set and so is $\mathcal{P}(\mathbb{N})$, but $\mathcal{P}(\mathbb{N})$ is a "bigger" infinite set than $\mathbb{N}$? This is indeed the case. In fact:

Exercise 6.2.5. Prove that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \cdots$.

So, there are many, many different magnitudes of infinity. Following convention, we denote $|\mathbb{N}| = \aleph_0$, which is pronounced aleph-null, or aleph-naught, or aleph-zero. ($\aleph$ is the first character in the Hebrew alphabet.) So, analogous to the situation with finite sets, we would write

$$|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$$

Please note, this is purely notation. We're not actually exponentiating anything.[1]

Exercise 6.2.6. Prove the following:

(a) If E is the set of even natural numbers, then $E \subset \mathbb{N}$ and $|E| = |\mathbb{N}| = \aleph_0$.

(b) $E \subset \mathbb{N} \subset \mathbb{Z}$, and $|E| = |\mathbb{N}| = |\mathbb{Z}| = \aleph_0$.

(c) $E \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$, and $|E| = |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0$.

(d) $E \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, and $|E| = |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0 < |\mathbb{R}|$. Wait, what?

[1] OK. We might be, but let's not get into that.

6.3 Binomial Coefficients

Factorials

Factorials arise so often in mathematics, that they deserve some special mention here. If $n \in \mathbb{N}$, then $n!$, pronounced "n factorial," is the natural number defined as follows:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1) \cdots 2 \cdot 1 & \text{if } n \geq 1 \end{cases}$$

Exercise 6.3.1. Compute the following:

(a) $0!$

(b) $1!$

(c) $2!$

(d) $3!$

(e) $4!$

(f) $5!$

(g) $6!$

(h) $7!$

Exercise 6.3.2. Prove that $n! = n \cdot (n-1)!$.

Binomial Coefficients

Given natural numbers $n, k \in \mathbb{N}$ with $n \geq k$, the **binomial coefficient** $\binom{n}{k}$ is the natural number

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

If either $k > n$ or either of n or k is negative, then we interpret $\binom{n}{k}$ to mean zero.

Exercise 6.3.3. Compute the following:

(a) $\binom{3}{1}$

(b) $\binom{4}{1}$

(c) $\binom{4}{2}$

(d) $\binom{5}{5}$

(e) $\binom{7}{3}$

(f) $\binom{10}{8}$

(g) $\binom{10}{11}$

(h) $\binom{10}{-1}$

(i) $\binom{10}{10}$

Prove this theorem.

Theorem 6.3.4 (Pascal's Rule)**.** If $n, k \in \mathbb{N}$, then $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

Exercise 6.3.5. Let $x \in \mathbb{R}$. Prove that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ as follows.

(a) Show the statement is true when $n = 0$.

(b) Show the statement is true when $n = 1$.

(c) And just for good measure, show the statement is true for $n = 2$ and $n = 3$.

(d) Ample base cases having been established, we'll continue by induction. Assume the statement is true for $n = m$. Write out this inductive hypothesis explicitly.

(e) Write out the statement we want to show is true in order to complete the inductive step.

(f) Express $(1+x)^{m+1}$ in a form where the inductive hypothesis can be used.

(g) Use the inductive hypothesis to replace the $(1+x)^m$ in the previous expression with a sum.

(h) Distribute.

(i) Split the sum into two sums.

(j) Shift the index of summation on the sum with the x^{k+1} in it so that the exponent on x is k. (i.e. Replace k with $k-1$. Pay attention to what happens to the summation indices.)

(k) That sum you just fiddled with should now start with $k=1$ and end with $k=m+1$, and the other sum starts with $k=0$ and ends with $k=m$. We want these to match. Observe that you can lower the lower bound of summation on the former sum to 0 without affecting the sum, and similarly you can raise the upper bound of summation on the latter sum to $m+1$ without affecting the sum. OK. So do that and write the new expression.

(l) Recombine the sums to a single sum.

(m) Apply Pascal's Rule.

(n) QED.

Exercise 6.3.6. Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Exercise 6.3.7. Prove that $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Exercise 6.3.8. Prove the Chu-Vandermonde Identity: $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r}\binom{n}{r-k}$.
(*Hint*: $(1+x)^{m+n} = (1+x)^m(1+x)^n$. Expand and equate coefficients.)

6.4 Binomial Coefficients, Fibonacci Numbers

Binomial Coefficients

Prove this theorem.

Theorem 6.4.1 (Hockey Stick Theorem)**.** If $n, r \in \mathbb{N}$, then

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

(*Hint*: Fix r and proceed by induction using the base case $n = r$.)

Exercise 6.4.2. Draw Pascal's triangle and figure out why the previous theorem is called the Hockey Stick Theorem.

Fibonacci Numbers

The Fibonacci numbers are the sequence of number $\{f_n\}_{n=1}^{\infty}$ defined recursively as follows:

$$\begin{aligned} f_1 &= 1 \\ f_2 &= 1 \\ f_n &= f_{n-1} + f_{n-2} \text{ for } n \geq 2 \end{aligned}$$

Exercise 6.4.3. Compute f_n for $n = 1, 2, 3 \ldots, 10$

Exercise 6.4.4. Prove that $f_1 - f_2 + f_3 - f_4 + \cdots - f_{2n-2} + f_{2n-1} = f_{2n} - 1$ for all natural numbers $n \geq 2$.

Exercise 6.4.5. Prove that $\sum_{i=1}^{n} f_n^2 = f_n \cdot f_{n+1}$ for all natural numbers $n \geq 1$.

Exercise 6.4.6. Prove that $f_{n-1} \cdot f_{n+1} = f_n^2 + (-1)^n$ for all natural numbers $n \geq 2$.

6.5 More Induction Practice

Exercise 6.5.1. Prove that for all $n \in \mathbb{N}$ such that $n \geq 1$

$$\sum_{k=1}^{n} 2^k = 2^{n+1} - 2$$

Exercise 6.5.2. Prove that for all $n \in \mathbb{N}$ such that $n \geq 1$

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$

Exercise 6.5.3. Prove that for all $n \in \mathbb{N}$ such that $n \geq 0$, $8^n - 3^n$ is divisible by 5. (*Hint*: It may be helpful to add and subtract $3 \cdot 8^k$ at some point.)

Exercise 6.5.4. Prove that the statement $n + 1 < n$ satisfies the inductive step portion of a proof by induction for all $n \in \mathbb{N}$. Is there a base case which can be established? Explain.

Exercise 6.5.5. Prove that $2^n < n!$ for all $n \in \mathbb{N}$ such that $n \geq 4$.

Exercise 6.5.6. Recall from calculus that if f_1 and f_2 are differentiable functions, then the product rule states that

$$(f_1 f_2)' = f_1' f_2 + f_1 f_2'$$

Generalize the product rule to the product of n functions. Prove this generalized product rule for all $n \in \mathbb{N}$ such that $n \geq 1$.

Exercise 6.5.7. Prove that for every $n \in \mathbb{Z}$ that $n(n^2 + 5)$ is a multiple of 6. (*Hint*: Split this into two cases: $n \geq 0$ and $n \leq 0$ and prove each case by induction.)

6.6 Still More Induction Practice

Intervals

A subset $I \subset \mathbb{R}$ is called an **interval** if whenever $a, b \in I$ and $c \in \mathbb{R}$ is so that $a < c < b$, then $c \in I$.

Exercise 6.6.1. Determine, with proof, which of the following are intervals.

(a) $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$

(b) $(0, 1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$

(c) $[-\pi, \infty) = \{x \in \mathbb{R} \mid x \geq -\pi\}$

(d) $\{1, 2, 3\}$

(e) $\varnothing$

(f) $\mathbb{R}$

Exercise 6.6.2. Prove the following.

(a) If I and J are intervals, then $I \cap J$ is an interval.

(b) The intersection of any finite number of intervals is an interval.

Exercise 6.6.3. Show that the statements in the previous exercise are false if "intersection" is replaced with "union."

Bounded Sets

A subset $X \subset \mathbb{R}$ is called **bounded** if there exists a real number $M \in \mathbb{R}$ so that $|x| < M$ for every $x \in X$.

Exercise 6.6.4. Prove that the union of any finite number of bounded sets is bounded.

Exercise 6.6.5. Prove that the intersection of any finite number of bounded sets is bounded.

$\mathbb{N}$

Recall that the successor of a set A is the set $S(A) = A \cup \{A\}$, and the the natural numbers $\mathbb{N}$ is the set containing $0 = \varnothing$ and all successors of 0.

Exercise 6.6.6. (a) Write out explicitly the sets 0, 1, 2, and 3 as sets of sets (of sets of...).

(b) Determine the number of $\varnothing$'s which appear in the set n. Prove your result.

(c) Determine the number of $\{$'s which appear in the set n. Prove your result.

(d) Determine the number of commas which appear in the set n. Prove your result.