MATH 241
Calculus & Analytic Geometry III

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Contents

Introduction

This document is a compilation of worksheets I have created for use in a one semester multivariable calculus course. My course met five days a week for 50 minutes per class. The weekly routine was (roughly) worksheet, lecture, computer lab, worksheet, lecture. That is, the worksheets were separated by class periods during which I lectured. Below, if a worksheet seems to be assuming knowledge of content which has not been covered by previous worksheets, it is likely that this content was covered in lecture. Also, each worksheet reference sections of a textbook. For my class, we used "Worldwide Multivariable Calculus" by David B. Massey and published by Worldwide Center of Mathematics, and all section references below are to this textbook. Of course, these worksheets can be useful in this study of calculus in the absence of this, or any, textbook.

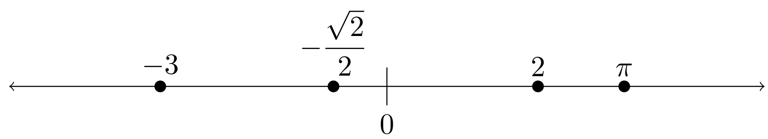
If you notice any typos or other errors, or are interested in my LaTeX code, please send me an email: wkronholm@whittier.edu. I would also appreciate hearing from instructors who use these materials in their courses.

1 Euclidean Space and Vectors

This worksheet discusses material corresponding roughly to sections 1.1 - 1.3 of your textbook.

Euclidean Spaces

The set $\mathbb{R}$ of real numbers should already be familiar to you. Sometimes we use interval notation and write $\mathbb{R} = (-\infty, \infty)$. We can picture $\mathbb{R}$ as a "number line" in the usual way:



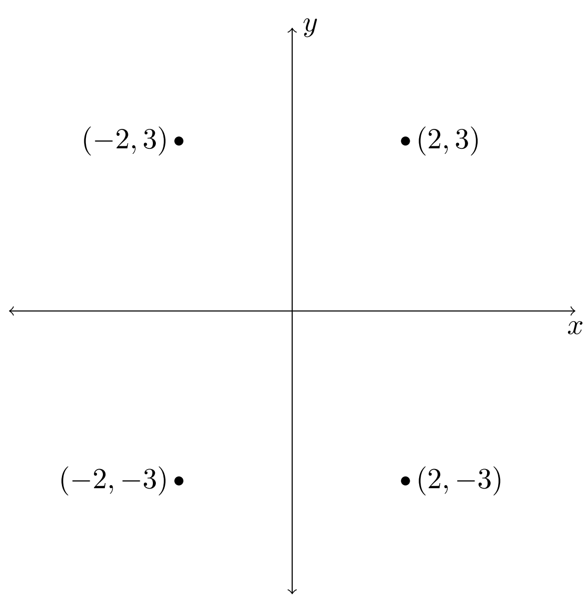
The elements of $\mathbb{R}$ are the real numbers. That is, 4, π, $\sqrt{1010}$, 0, and -12.76 are all real numbers while $\sqrt{-1}$, $\frac{1}{0}$, ∞, and $\int$ are not. We would write this with the following mathematical notation:

$$\pi \in \mathbb{R}$$

$$\sqrt{-1} \notin \mathbb{R}$$

You should read the first statement as "π is in $\mathbb{R}$" or "π is a member of the set of all real numbers" or "π is an element of the set $\mathbb{R}$." The second statement can be read as "$\sqrt{-1}$ is not a real number."

Also familiar to you is $\mathbb{R}^2$, the set of ordered pairs of real numbers. These are pictured in the plane in the usual way.



Again, we would write things like $(-\pi, \sqrt{2}) \in \mathbb{R}^2$ and $4 \notin \mathbb{R}^2$.

Of course, there is no reason to stop with ordered pairs. We can just as easily discuss ordered triples, ordered quadruples, etc., of real numbers. For example, we have the following:

$$(2, 5, 7) \in \mathbb{R}^3$$

$$(0, -1, 3.2, 7.92) \in \mathbb{R}^4$$

$$(0, 0, 0, 0, 0, 0, 1, 0, 0, 0) \in \mathbb{R}^{10}$$

$$(\pi, -\sqrt{2}, 0, -1) \in \mathbb{R}^4$$

Each of $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\ldots$, is a *Euclidean space.* A generic Euclidean space is denoted $\mathbb{R}^n$ and is said to have *dimension* n.

Exercise 1. Determine the dimension of the Euclidean space which contains the following points. Indicate this by writing adding the symbols $\in \mathbb{R}^n$ to the right of each point with the appropriate value of n.

(a) $(1, 2, 3)$

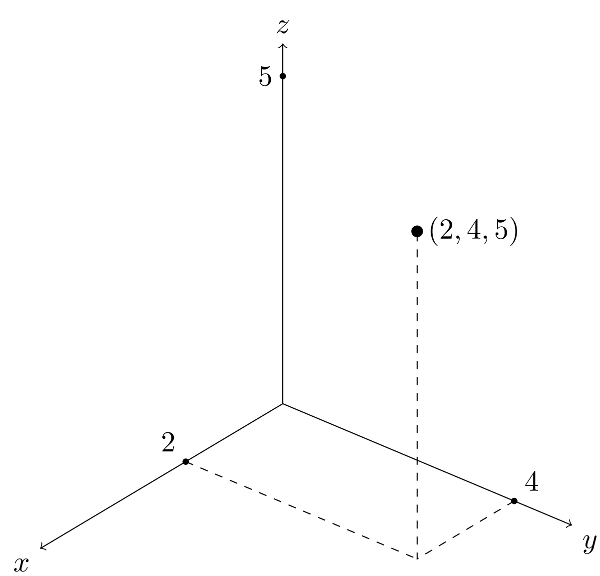
(b) $(0, -4, -7, \pi/2, 1)$

(c) $(\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \pi)$

(d) $(1, 1.1, 1.11, 1.111, 1.1111, 1.11111, 1.111111, 1.1111111)$

(e) $(2.98498874385093092, -6.2984287875387532987532879532 87)$

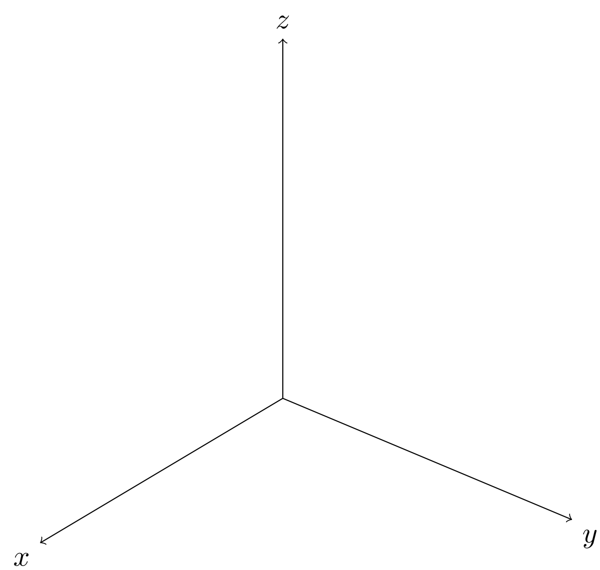
Plotting the points in $\mathbb{R}^n$ is difficult to visualize when $n \geq 4$. But in $\mathbb{R}^3$, it's not so bad. Consider the point $(2, 4, 5) \in \mathbb{R}^3$.



One way to think of this is that the x- and y-axes form the horizontal plane and the z-axis is vertical. There are other choices we could have made, and we'll get to that in a bit.

Exercise 2. Plot the following points in $\mathbb{R}^3$ on the axes below.

- $(1, 2, 3)$
- $(-1, 0, 4)$
- $(2, -1, 3)$
- $(0, 3, 0)$



We'll want to consider distances in Euclidean spaces as well. This is not so bad, once we look for a reasonable pattern.

Exercise 3. Draw a rectangle with side lengths 2 and 3. Find the length of the diagonal.

Exercise 4. Draw a rectangular box with side lengths 2, 3, and 4. Find the length of the diagonal.

Exercise 5. Find the distance between the points $(2, 3)$ and $(1, 5)$ in $\mathbb{R}^2$.

Exercise 6. Find the distance between the points $(1, 2, 3)$ and $(0, 5, 5)$ in $\mathbb{R}^3$.

Exercise 7. Find the distance between the points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$.

Exercise 8. Find the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbb{R}^3$.

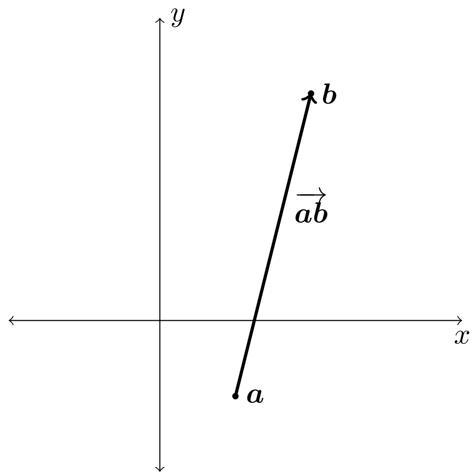
Exercise 9. Find the distance between the points (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) in $\mathbb{R}^4$.

Exercise 10. Generalize. That is, find the distance between the points $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$.

Vectors

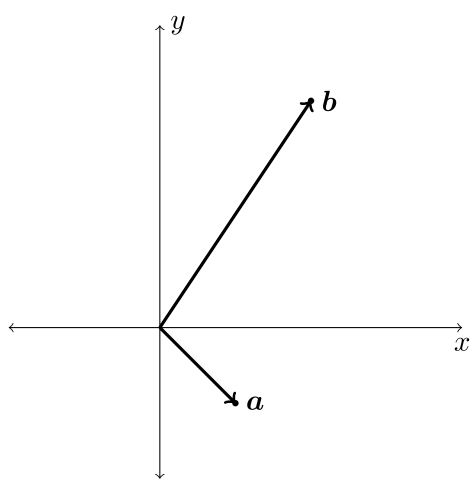
You may have been told in a physics class or somewhere else that a *vector* is a direction and a magnitude. The definition works intuitively enough. We'll be thinking of vectors in Euclidean space as an arrow that points from one point to another.

For example, consider the points $\boldsymbol{a} = (1, -1)$ and $\boldsymbol{b} = (2, 3)$. The the *displacement vector* from $\boldsymbol{a}$ to $\boldsymbol{b}$ is the vector $\overrightarrow{\boldsymbol{ab}}$ pictured below.

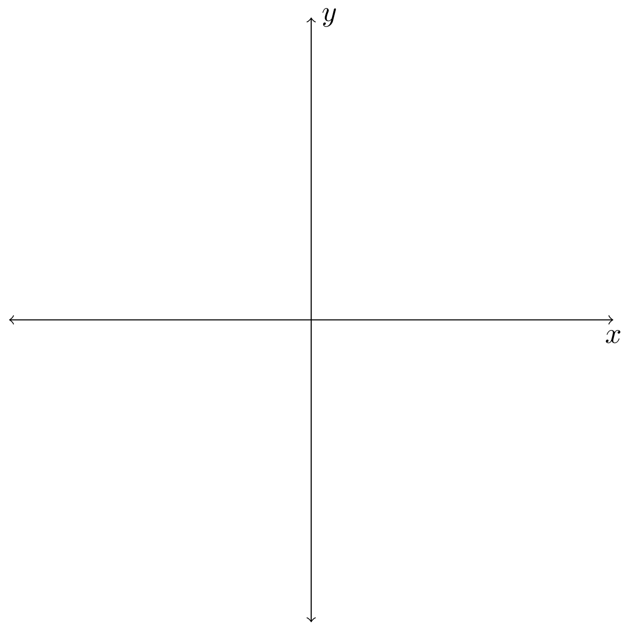


Exercise 11. What is the difference between $\overrightarrow{\boldsymbol{ab}}$ and $\overrightarrow{\boldsymbol{ba}}$?

Given any point, we can also think of it as a vector based at the origin. This means we draw an arrow from the origin to our point. For example, the points $\boldsymbol{a}$ and $\boldsymbol{b}$ above can also be thought of as the vectors drawn below.

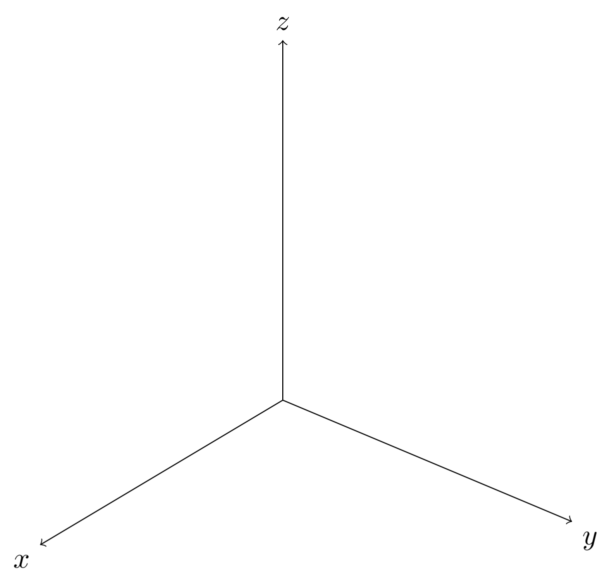


Exercise 12. For the points $\boldsymbol{a} = (-2, 4)$, $\boldsymbol{b} = (0, 5)$, and $\boldsymbol{c} = (0, -5)$ in $\mathbb{R}^2$, draw the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\overrightarrow{\boldsymbol{ca}}$, $\overrightarrow{\boldsymbol{ab}}$, and $\overrightarrow{\boldsymbol{bc}}$.



All the same comments apply to vectors in $\mathbb{R}^3$ as well.

Exercise 13. For the points $\boldsymbol{a} = (1, 2, 4)$, and $\boldsymbol{b} = (0, 3, 1)$ in $\mathbb{R}^3$, draw the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\overrightarrow{\boldsymbol{ab}}$.



Of course, all of the same comments apply to vectors in a general Euclidean space $\mathbb{R}^n$ as well. Those are just harder to draw when $n \geq 4$.

Magnitude and Direction

The *magnitude* of a vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ in $\mathbb{R}^n$ is denoted $|\boldsymbol{a}|$ and is the real number

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Magnitude is also sometimes called length because it is the length of the arrow from the origin to the point $\boldsymbol{a}$ in $\mathbb{R}^n$. Let's reiterate one point: the magnitude of a vector is a real number. That is:

$$\boldsymbol{a} \in \mathbb{R}^n$$

$$|\boldsymbol{a}| \in \mathbb{R}$$

Exercise 14. Find the magnitudes of the vectors in the previous exercise.

To talk about direction, we need to build up some more notions. It's not too difficult to define a notion of addition of vectors. If $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ are vectors in $\mathbb{R}^n$, then we get a new vector $\boldsymbol{a} + \boldsymbol{b}$ in $\mathbb{R}^n$ given by

$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

Exercise 15. Notice that $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$.

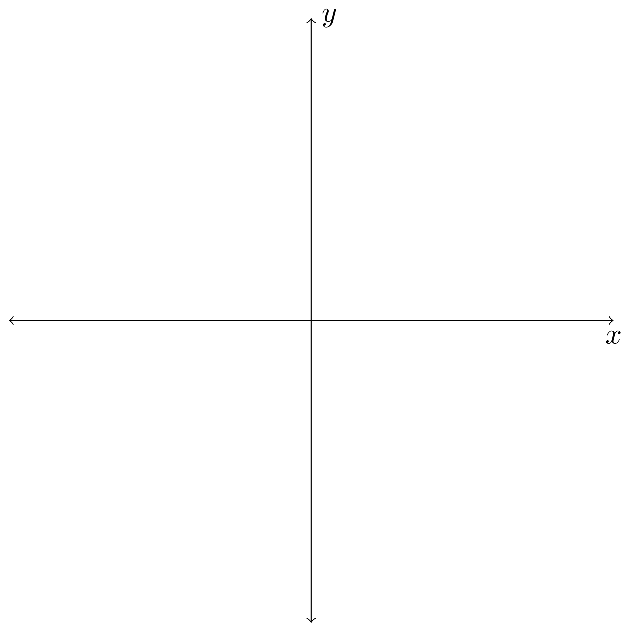
Exercise 16. Find the sum of each pair of vectors from Exercise 13.

We also have a special kind of multiplication. If $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ is a vector in $\mathbb{R}^n$ and r is a real number (called a *scalar*), the we can form the *scalar multiplication* of $\boldsymbol{a}$ by r to be the vector $r\boldsymbol{a}$ given by

$$r\boldsymbol{a} = (ra_1, ra_2, \ldots, ra_n)$$

If $r \neq 0$, then we will may write $\dfrac{\boldsymbol{a}}{r}$ instead of $\dfrac{1}{r}\boldsymbol{a}$. Also, we write $-\boldsymbol{a}$ in place of $(-1)\boldsymbol{a}$ and $\boldsymbol{b} - \boldsymbol{a}$ in place of $\boldsymbol{b} + (-1)\boldsymbol{a}$.

Exercise 17. For the points $\boldsymbol{a} = (-2, 4)$ and $\boldsymbol{b} = (0, 5)$ in $\mathbb{R}^2$, draw the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $2\boldsymbol{a}$, $-2\boldsymbol{a}$, $\frac{1}{2}\boldsymbol{a}$, $\overrightarrow{\boldsymbol{ab}}$, and $\boldsymbol{b} - \boldsymbol{a}$.



Exercise 18. Describe in words what happens to a vector $\boldsymbol{a}$ after scalar multiplication by the real number r.

Exercise 19. Consider the vector $\boldsymbol{a} = (2, 3)$ in $\mathbb{R}^2$. Compute $|\boldsymbol{a}|$, $\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$ and $\left|\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}\right|$.

Any vector with magnitude 1 is called a *unit vector*. Given any non-zero vector $\boldsymbol{a}$ in $\mathbb{R}^n$, the vector $\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$ is a unit vector and we call this vector the *direction* of the vector $\boldsymbol{a}$. That is, the direction of a vector is the unit vector which points in the same direction as the vector $\boldsymbol{a}$.

Exercise 20. Find the direction of each of the vectors below.

$\boldsymbol{a} = (2, 3)$

$\boldsymbol{b} = (-2, -3)$

$\boldsymbol{c} = (1, 2, -3)$

$\boldsymbol{d} = (0, 1, -1)$

$\boldsymbol{e}_1 = (1, 0, 0)$

$\boldsymbol{e}_2 = (0, 1, 0)$

$\boldsymbol{e}_3 = (0, 0, 1)$

2 Dot Product, Angles, and Orthogonal Projection

This worksheet discusses material corresponding roughly to section 1.3 of your textbook.

Dot product

An important product of vectors is the **dot product** (or **scalar product**, or **inner product**). If $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ are vectors in $\mathbb{R}^n$, then we define the **dot product** $\boldsymbol{a} \cdot \boldsymbol{b}$ to be $a_1b_1 + a_2b_2 + \cdots + a_nb_n$. The symbol $\cdot$ is commonly used for products of scalars, too. For example, you are probably in the habit of writing $4 \cdot 5 = 20$. The usage of $\cdot$ will be unambiguous from context. (i.e. If a and b are scalars, then $a \cdot b$ means "a times b." If $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, then $\boldsymbol{a} \cdot \boldsymbol{b}$ means "the dot product of $\boldsymbol{a}$ and $\boldsymbol{b}$. Actually, the usage is totally unambiguous, since scalars are the same as one-dimensional vectors ((or at least the set of scalars is isomorphic (((iso-what?))) to the set of one-dimensional vectors))).

Exercise 21. Let $\boldsymbol{v} = (2, -4, 3)$ and $\boldsymbol{w} = (-1, 2, 5)$ be vectors in $\mathbb{R}^3$. Compute $\boldsymbol{v} \cdot \boldsymbol{w}$ and $\boldsymbol{w} \cdot \boldsymbol{v}$.

Exercise 22. Convince yourself the following are true for all n-dimensional vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ and all scalars $c \in \mathbb{R}$.

1. $\boldsymbol{v} \cdot \boldsymbol{w}$ is a real number. (i.e. It's not a vector.)
2. $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.
3. $\boldsymbol{v} \cdot (\boldsymbol{w} + \boldsymbol{u}) = \boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{u}$.
4. $(c\boldsymbol{v}) \cdot \boldsymbol{w} = c(\boldsymbol{v} \cdot \boldsymbol{w})$.
5. $\boldsymbol{v} \cdot \boldsymbol{v} = |\boldsymbol{v}|^2$.
6. $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq |\boldsymbol{v}||\boldsymbol{w}|$.
7. $\boldsymbol{v} \cdot \boldsymbol{w} = |\boldsymbol{v}||\boldsymbol{w}| \cos(\theta)$ where θ is the **angle between $\boldsymbol{v}$ and $\boldsymbol{w}$**.

Exercise 23. Notice that property 7 above implies that $\boldsymbol{v} \neq \boldsymbol{0}$ and $\boldsymbol{w} \neq \boldsymbol{0}$, then $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ if and only if $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular. If this happens, we say that $\boldsymbol{v}$ and $\boldsymbol{w}$ are **orthogonal**. You can think of orthogonal as a synonym for perpendicular. Both words capture the idea of "meeting at a right angle."

Exercise 24. Consider the vectors $\boldsymbol{a} = (2, 1, 3)$, $\boldsymbol{b} = (0, 3, -1)$, and $\boldsymbol{c} = (3, 0, 0)$ in $\mathbb{R}^3$. For each pair of vectors, determine if the angle between them is acute, obtuse, or right.

There are some special vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ which we need to pay attention to.

Exercise 25. Let $\boldsymbol{i} = (1, 0)$ and $\boldsymbol{j} = (0, 1)$ be vectors in $\mathbb{R}^2$. Determine the magnitudes of $\boldsymbol{i}$ and $\boldsymbol{j}$ and determine the angle between them. Why are these special? (*Hint:* Plot them in the plane and observe.)

Exercise 26. Let $\boldsymbol{i} = (1, 0, 0)$, $\boldsymbol{j} = (0, 1, 0)$, and $\boldsymbol{k} = (0, 0, 1)$ be vectors in $\mathbb{R}^3$. Determine the magnitudes of $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ and determine the angles between each pair of them. Why are these special? (*Hint:* Plot them in space and observe.)

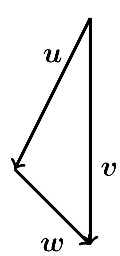
Exercise 27. Let $\boldsymbol{a} = (a_1, a_2, a_3)$ be an arbitrary vector in $\mathbb{R}^3$. Find scalars r, s, and t so that $\boldsymbol{a} = r\boldsymbol{i} + s\boldsymbol{j} + t\boldsymbol{k}$ where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the same vectors as in the previous exercise. Now why are the vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ special?

Orthogonal Projection

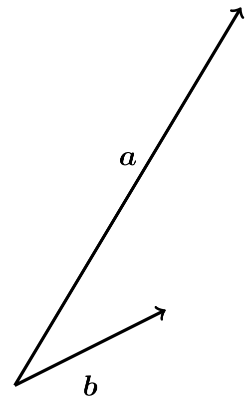
Suppose we have two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$. We'll want to make sense of "the part of $\boldsymbol{a}$ which is parallel to $\boldsymbol{b}$." To help us, first we'll think a geometrically about vector addition.

Exercise 28. Consider the vectors $\boldsymbol{a} = (2, 1)$ and $\boldsymbol{b} = (1, 3)$ in $\mathbb{R}^2$. Draw the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{a} + \boldsymbol{b}$ in the plane. These vectors are related to a parallelogram. Determine the parallelogram and explain the relationship.

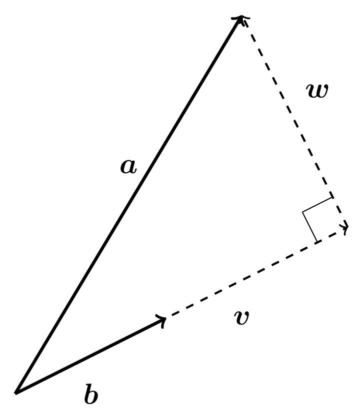
Exercise 29. Consider the vectors $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{u}$ in the picture below. Write down an equation which relates the three vectors together.



Back to projection. Consider the picture below.



We want to make sense of the notion "the part of $\boldsymbol{a}$ which is parallel to $\boldsymbol{b}$." To do this, we want to be able to write $\boldsymbol{a} = \boldsymbol{v} + \boldsymbol{w}$ where $\boldsymbol{v}$ is parallel to $\boldsymbol{b}$ and $\boldsymbol{w}$ is orthogonal to $\boldsymbol{b}$. That is, we want to find vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ as in the picture below.



Since $\boldsymbol{v}$ is parallel to $\boldsymbol{b}$, there must be some real number t so that $\boldsymbol{v} = t\boldsymbol{b}$. We need to determine this number t.

Exercise 30. You'll determine a formula for t based on $\boldsymbol{a}$ and $\boldsymbol{b}$ as follows.

(a) Write down a vector equation that relates $\boldsymbol{a}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

(b) Replace $\boldsymbol{v}$ with $t\boldsymbol{b}$.

(c) Dot both sides of the equation with $\boldsymbol{b}$.

(d) Simplify. (*Hint:* What is $\boldsymbol{b} \cdot \boldsymbol{w}$? Why?)

(e) Solve for t.

Your work above allows us to make the following definition: The **orthogonal projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$** is the vector

$$\mathrm{proj}_{\boldsymbol{b}}\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b}$$

Exercise 31. Find the orthogonal projection of $(-1, 2)$ onto $(4, 3)$. Sketch a picture illustrating the projection.

Exercise 32. Find the orthogonal projection of $(0, 1, 2)$ onto $(-1, 2, 3)$. Sketch a picture illustrating the projection.

Exercise 33. Find the projection of $\boldsymbol{a} = (a_1, a_2)$ onto each of $\boldsymbol{i}$ and $\boldsymbol{j}$.

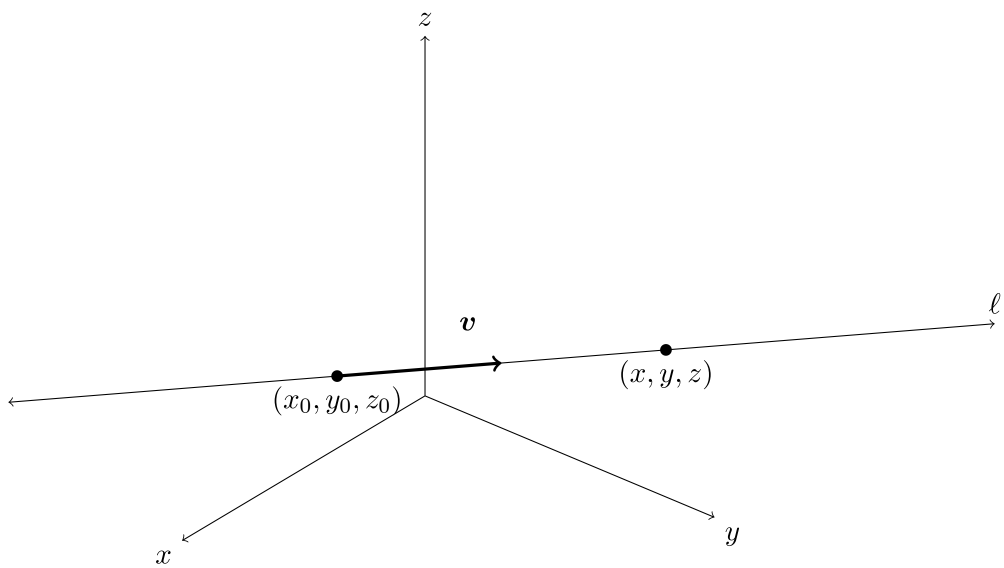
Exercise 34. Find the projection of $\boldsymbol{a} = (a_1, a_2, a_3)$ onto each of $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$.

3 Lines and Planes in $\mathbb{R}^3$

This worksheet discusses material corresponding roughly to section 1.4 of your textbook.

Lines in $\mathbb{R}^3$

Just as in $\mathbb{R}^2$, a line in $\mathbb{R}^3$ is determined by two pieces of information: a point (x_0, y_0, z_0) and a direction vector $\boldsymbol{v} = (v_1, v_2, v_3)$. Thus, we have a picture like the followign.



Any point (x, y, z) on this line must be some multiple of $\boldsymbol{v}$ "away from" the point (x_0, y_0, z_0). More precisely, the displacement vector from (x_0, y_0, z_0) to (x, y, z) must be a scalar t times $\boldsymbol{v}$.

Exercise 35. (a) Write an equation which capture the notion "the displacement vector from (x_0, y_0, z_0) to (x, y, z) is a scalar t times $\boldsymbol{v}$."

(b) Solve your equation for (x, y, z). This is the **vector equation** for the line.

(c) Find the vector equation for the line through the point $(2, 6, -1)$ in the direction of the vector $\boldsymbol{v} = (0, 3, 5)$.

(d) Find the vector equation for the line which passes through $(-3, 2, 7)$ and also the point $(3, 1, 6)$.

Exercise 36. (a) Write again the vector equation for the line through the point (x_0, y_0, z_0) in the direction of the vector $\boldsymbol{v} = (v_1, v_2, v_3)$.

(b) Write out three equations, one for x, one for y, and one for z. That is, complete the following:

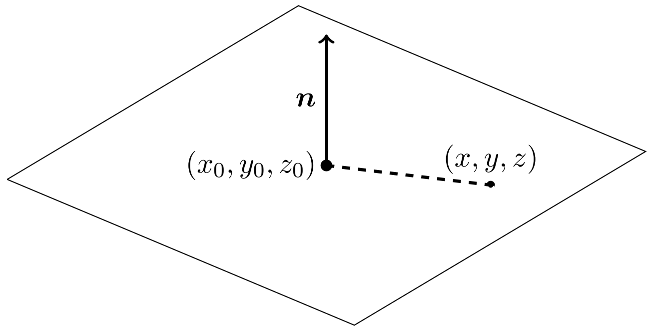
$$\begin{aligned} x(t) &= \\ y(t) &= \\ z(t) &= \end{aligned}$$

(c) Declare the set of equations you wrote to be the **parametric equations** for the line.

(d) Determine the parametric equations for the lines you found above in Exercises 35(c) and 35(d).

Planes in $\mathbb{R}^3$

In $\mathbb{R}^2$ a line is determined by a point and a normal direction to the line. In $\mathbb{R}^3$, a plane is determined by a point and a normal direction for the plane. Consider the picture below.



If $\boldsymbol{n} = (a, b, c)$ is a normal vector for the plane and (x_0, y_0, z_0) is a point on the plane, then we can derive an equation for the plane as follows.

Exercise 37. Let (x, y, z) be another point on the plane.

(a) Compute the displacement vector from (x_0, y_0, z_0) to (x, y, z).

(b) Write an equation which says that this displacement vector is orthogonal to the normal vector $\boldsymbol{n} = (a, b, c)$.

(c) Expand the equation you have above. (That is, compute the dot product of the two vectors.)

(d) Observe that your equation is of the form $ax + by + cz + d = 0$.

Exercise 38. Find the normal vector to the plane $3x - 2y + z - 6 = 0$.

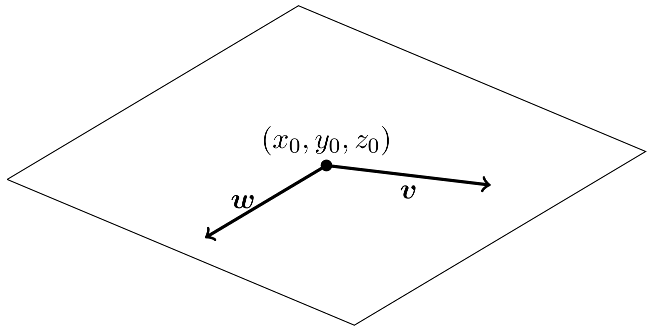
Exercise 39. Find the equation of the plane in $\mathbb{R}^3$ which passes through the point $(2, -3, 2)$ and which is orthogonal to the vector $(1, -1, 8)$.

Exercise 40. Is the point $(2, 3, 4)$ on the plane $3x - 2y = 0$? What about the point $(1, 1, 1)$?

Exercise 41. Find the equation of the plane which is parallel to the plane $x + y - z = 10$ and which passes through the point $(0, 1, 2)$.

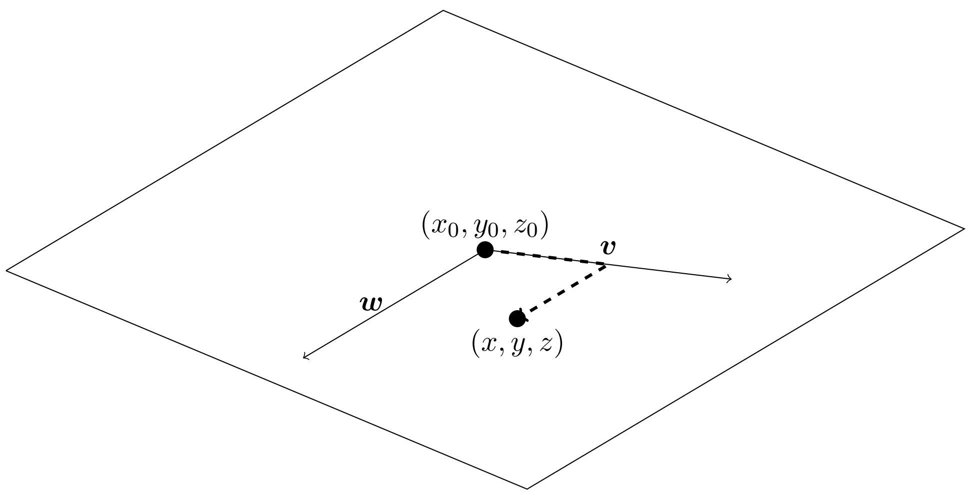
Exercise 42. Find equations for the xy-plane, the xz-plane, and the yz-plane.

We don't always know a normal direction for the plane we are interested in. Sometimes we instead know two non-parallel directions which keep us in the plane and a point in the plane. Consider the picture below.



In this picture, the point (x_0, y_0, z_0) is in the plane, and the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are both parallel to the plane, but are not parallel to each other.

If (x, y, z) is any point in the plane, then we can get to (x, y, z) from $x_0, y_0, z_0)$ by taking a step in the $\boldsymbol{v}$ direction, followed by a step in the $\boldsymbol{w}$ direction. Consider the augmented picture below.



Thus, and point (x, y, z) in the plane can be described by the **vector equation** for the plane

$$(x, y, z) = (x_0, y_0, z_0) + t\boldsymbol{v} + s\boldsymbol{w}$$

where s and t are **parameters**.

Just as with lines, the vector equation can be exploded to give **parametric equations** for the plane.

Exercise 43. Determine the parametric equations for the plane above. That is, complete the equations below:

$$\begin{aligned} x(s,t) &= \\ y(s,t) &= \\ z(s,t) &= \end{aligned}$$

Exercise 44. Find the vector and parametric equations for the plane which passes through the point $(1, 2, 3)$ and is parallel to the vectors $(1, 0, 1)$ and $(1, 2, 0)$.

Exercise 45. Find the vector and parametric equations for the plane which passes through the points $(0, 1, 2)$, $(5, -2, 3)$ and $(0, 1, 0)$.

Exercise 46. Find a vector equation and parametric equations for the plane $2x + y - z - 3 = 0$.

4 The Cross Product in $\mathbb{R}^3$ and Functions of a Single Real Variable

This worksheet discusses material corresponding roughly to sections 1.5-1.6 of your textbook.

The Cross Product

Recall that if $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ are vectors in $\mathbb{R}^3$, then the **cross product** $\boldsymbol{a} \times \boldsymbol{b}$ is the vector given by the formula below:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}$$

Remember, the determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The cross product $\boldsymbol{a} \times \boldsymbol{b}$ is a vector which is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$, has magnitude equal to the area of the parallelogram determined by $\boldsymbol{a}$ and $\boldsymbol{b}$, and is directed so that $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{a} \times \boldsymbol{b}$ form a right-handed system.

Exercise 47. Let $\boldsymbol{a} = (0, 2, 1)$ and $\boldsymbol{b} = (-1, -2, 0)$. Compute $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{b} \times \boldsymbol{a}$. What do you notice?

Exercise 48. Compute all six possible cross products by taking pairs of vectors from among $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$.

Exercise 49. Find a vector which is orthogonal to both $(-2, 3, 1)$ and $(0, 5, -2)$.

Exercise 50. Find a normal vector for the plane which contains the points $(1, 1, 1)$, $(2, 1, 1)$, and $(0, -2, -1)$. Then write the standard equation for the plane.

Exercise 51. Find the standard equation for the plane which has the following parametric equations:

$$\begin{aligned} x(s,t) &= 2+3s-t \\ y(s,t) &= s+t \\ z(s,t) &= s-t \end{aligned}$$

Functions of a Single Real Variable

We now shift gears and discuss functions, which are the main objects of study in any calculus class. To start, we consider functions of a single real variable. These are functions $f\colon \mathbb{R} \to \mathbb{R}^n$, or if A is a subset of the real numbers, then we'll also consider functions $f\colon A \to \mathbb{R}^n$. The domain of functions of a single real variable is a subset of $\mathbb{R}$ and the codomain is a Euclidean space $\mathbb{R}^n$. (Notice that if $n = 1$ then these are the functions you typically considered in algebra and the first year of calculus.

Exercise 52. Consider the function $f\colon [0,\pi] \to \mathbb{R}^2$ given by $f(t) = (\cos(t), \sin(t))$. Compute $f(0)$, $f(\pi/6)$, $f(\pi/4)$, $f(\pi/2)$, $f(3\pi/4)$, $f(2\pi/3)$, $f(\pi)$, and $f(5\pi/2)$. Sketch a graph of the image of f in $\mathbb{R}^2$.

Exercise 53. Consider the function $f\colon [0,\pi] \to \mathbb{R}^2$ given by $f(t) = (\cos(\pi - t), \sin(\pi - t))$. Compute $f(0)$, $f(\pi/6)$, $f(\pi/4)$, $f(\pi/2)$, $f(3\pi/4)$, $f(2\pi/3)$, $f(\pi)$, and $f(5\pi/2)$. Sketch a graph of the image of f in $\mathbb{R}^2$.

Exercise 54. Consider the function $g\colon \mathbb{R} \to \mathbb{R}^3$ given by $g(t) = (\cos(t), \sin(t), t)$. Sketch a graph of the image of g in $\mathbb{R}^3$.

Any function $f\colon \mathbb{R} \to \mathbb{R}^n$ can be written as $f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$ where each $f_i\colon \mathbb{R} \to \mathbb{R}$ is a real valued function. These functions f_i are called the **component functions** of f.

Exercise 55. Determine the component functions for the functions in the previous three exercises.

We say that a function $f\colon \mathbb{R} \to \mathbb{R}^n$ is **continuous** if each of its component functions is continuous. Similarly, we say a function $f\colon \mathbb{R} \to \mathbb{R}^n$ is **differentiable** if each of its component functions is differentiable. In the case of a differentiable function, we have the following:

$$f'(t) = (f_1'(t), f_2'(t), \ldots, f_3'(t))$$

Exercise 56. Compute the derivatives of the functions below.

(a) $f(t) = (t^2, e^t, \tan(t))$

(b) $g(t) = (\sqrt{t^2+1}, 10t - 4)$

(c) $h(t) = \left(t\sqrt{t^2+1}, t^5, t^{7/9}, \ln(t), \dfrac{t}{t^2+1}\right)$

If $p\colon \mathbb{R} \to \mathbb{R}^n$ is the position of an object in Euclidean space at time t, then the **velocity** of the object is $v(t) = p'(t)$, the **speed** of the object is $|v(t)|$, and the **acceleration** of the object is $a(t) = v'(t) = p''(t)$.

Exercise 57. Suppose an object is moving in the plane with a position of $p(t) = (\cos(t), \sin(t))$. Determine the velocity, speed, and acceleration of the object. Also, determine the angle between the velocity and acceleration vectors at time $t = 3\pi/4$. Sketch a picture including the image of the position curve in $\mathbb{R}^2$ as well as the velocity and acceleration vectors at time $t = 3\pi/4$.

Exercise 58. Suppose an object is moving in the space with a position of $p(t) = (\cos(t), \sin(t), t)$. Determine the velocity, speed, and acceleration of the object.

5 Multivariable Functions and Partial Derivatives

This worksheet discusses material corresponding roughly to sections 1.7 and 2.1 of your textbook.

Multivariable Functions

We now consider functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$. To begin, we'll restrict to functions $f\colon \mathbb{R}^n \to \mathbb{R}$. As is typical, unless otherwise stated, the domains of our functions will be assumed to be as large as possible.

Exercise 59. Consider the functions below. For each one, determine the domain.

(a) $f(x, y) = x^2 + y^3$

(b) $g(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$

(c) $h(x, y, z, w) = \ln(x) + yzw$

(d) $i(x, y) = \dfrac{1}{xy}$

(e) $j(x, y) = \sqrt{xy}$

(f) $k(x, y) = \sin(xy) + e^{x^2+y^2} - xy^3 + 3$

Exercise 60. For the same functions as in the previous exercise, compute:

(a) $f(3, -1)$

(b) $g(0, 0, 0)$

(c) $h(e, 1, -1, -1)$

(d) $i(-2, 1/2)$

(e) $j(-6, -8)$

(f) $k(0, 0)$

Recall that **elementary functions** are those which are constant functions, power functions, polynomial functions, exponential functions, logarithmic functions, trigonometric functions, inverse trigonometric functions, or finite combinations of these using addition, subtraction, multiplication, division, and composition.

Exercise 61. Determine which of the functions in Exercise 59 are elementary functions.

As in single variable calculus, all elementary functions are continuous on their domains.

Exercise 62. Determine where the functions in Exercise 59 are continuous.

Partial Derivatives

Let's consider a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2y + x^3y^2$. Thus f is a real valued function of two real variables. If we hold one of the variables, say y, constant, then we can view f as a function of x only, and we can differentiate f with respect to x and obtain $2xy + 3x^2y^2$. Similarly, if we hold x constant, then we can view f as a function of y only and differentiate with respect to y to obtain $x^2 + 2x^3y$. These are the **partial derivatives** of f.

Here's a definition:

Definition 1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function and write $z = f(x, y)$. Then the **partial derivative of f with respect to x** is the function

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

provided the limit exists. The following notations all denote the partial derivative of f with respect to x:

$$f_x(x, y),\ f_1(x, y),\ \frac{\partial f}{\partial x},\ \frac{\partial z}{\partial x}$$

Similarly, we define the **partial derivative of f with respect to y** to be

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

provided the limit exists. The following notations all denote the partial derivative of f with respect to y:

$$f_y(x, y),\ f_2(x, y),\ \frac{\partial f}{\partial y},\ \frac{\partial z}{\partial y}$$

The operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are called the **partial differential operators**.

Exercise 63. Apply $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ to each function below.

(a) $f(x, y) = x^2 + y^3 + 12$

(b) $g(x, y) = \dfrac{1}{x^2 + y^2}$

(c) $h(x, y) = \ln(x) + \arctan(y)$

(d) $i(x, y) = \dfrac{1}{xy}$

(e) $j(x, y) = \sqrt{xy}$

(f) $k(x, y) = \sin(xy) + e^{x^2+y^2} - xy^3 + 3$

Of course, we can compute higher order partial derivatives by applying each of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ multiple times. The total number of times you take a partial derivative is called the **order** of the partial derivative.

For example, if $z = f(x, y)$ is a real valued function, then we have the first order partial derivatives

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$$

and the second order partial derivatives

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial x}, \frac{\partial}{\partial x}\frac{\partial f}{\partial y}, \frac{\partial}{\partial y}\frac{\partial f}{\partial x}, \frac{\partial}{\partial y}\frac{\partial f}{\partial y}$$

We introduce a bit more notation:

$$\begin{aligned}
f_{xx}(x, y) &= \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} \\
f_{xy}(x, y) &= \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} \\
f_{yx}(x, y) &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \\
f_{yy}(x, y) &= \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

Notice that variables the subscripts on f appear typographically in the opposite order to the partial differential operators. This is because you are to interpret f_{xy} to mean "first differentiate f with respect to x, then differentiate with respect to y."

Exercise 64. Let $f(x, y) = x^2 y + xy^3 - 3xy + 7$. Compute f_{xx}, f_{xy}, f_{yx}, and f_{yy}.

Exercise 65. Let $h(x, y) = \ln(x) + \arctan(y)$. Compute h_{xx}, h_{xy}, h_{yx}, and h_{yy}.

Exercise 66. Let $g(x,y) = \dfrac{1}{x^2+y^2}$. Compute g_{xx}, g_{xy}, g_{yx}, and g_{yy}.

You probably noticed that in the exercises above, $f_{xy} = f_{yx}$, $g_{xy} = g_{yx}$, and $h_{xy} = h_{yx}$. This doesn't always happen, but "most" of the time it does. The precise statement is below.

Theorem. Suppose f_x, f_y and f_{xy} exist in an open ball centered at (x_0, y_0), and that f_{xy} is continuous at (x_0, y_0). Then $f_{yx}(x_0, y_0)$ exists and

$$f_{xy}(x_o, y_0) = f_{yx}(x_0, y_0).$$

Thus, when we work with "nice" functions, the order in which we take partial derivatives is irrelevant. It turns out that elementary functions are nice.

Exercise 67. Let $f(x,y) = x^2y + \dfrac{\sin(y^3)e^{3y+1}}{\arccos(\arctan(\ln(y)))}$. Compute $\dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y}$.

Exercise 68. Suppose $f(x, y)$ is "nice" so that all partial derivatives of all orders exist and are continuous.

(a) How many first order partial derivatives does f have?

(b) How many second order partial derivatives does f have? (Remember, our assumptions are such that $f_{xy} = f_{yx}$, so this only counts as one function.)

(c) How many third order partial derivatives does f have?

(d) How many fourth order partial derivatives does f have?

Exercise 69. Repeat the previous exercise, but dropping the assumption that f is "nice." That is, do not assume $f_{xy} = f_{yx}$ and count the number of first, second, third, and fourth order partial derivatives of f.

6 Differentiation Rules!

This worksheet discusses material corresponding roughly to section 2.4 of your textbook.

Just like in single variable calculus, there are some handy rules for computing derivatives of certain combinations of multivariable functions. We begin with the simplest case.

Linearity of the Derivative

Let f and g be real valued functions defined on a subset of $\mathbb{R}^n$, each of which is differentiable at p. Then for any real numbers a and b

$$\vec{\nabla}(af + bg)(p) = a\vec{\nabla} f(p) + b\vec{\nabla} g(p).$$

Exercise 70. Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}$ and $g\colon \mathbb{R}^3 \to \mathbb{R}$ are differentiable at $(2, 3, 4)$, $\vec{\nabla} f(2, 3, 4) = (6, 1, 0)$, and $\vec{\nabla} g(2, 3, 4) = (-2, 0, 1)$. Compute $\vec{\nabla}(-2f + 4g)(2, 3, 4)$

Exercise 71. Let $f(x, y) = 3x^2y$ and $g(x, y) = \sin(xy) + e^{xy}$. Compute $\vec{\nabla}(2f - g)$.

Product Rule

Let f and g be real valued functions defined on a subset of $\mathbb{R}^n$. If f and g are differentiable at p, then fg is also differentiable at p and

$$\overrightarrow{\nabla}(fg)(p) = g(p)\overrightarrow{\nabla}f(p) + f(p)\overrightarrow{\nabla}g(p).$$

That is,

$$\overrightarrow{\nabla}(fg) = g\overrightarrow{\nabla}(f) + f\overrightarrow{\nabla}(g).$$

Exercise 72. Compute $\overrightarrow{\nabla}[(xy + \tan(x))(\ln(y) - x^3y)]$

Quotient Rule

Let f and g be real valued functions defined on a subset of $\mathbb{R}^n$. If f and g are differentiable at p and $g(p) \neq 0$, then

$$\overrightarrow{\nabla}\left(\frac{f}{g}\right)(p) = \frac{g(p)\overrightarrow{\nabla}f(p) - f(p)\overrightarrow{\nabla}g(p)}{(g(p))^2}.$$

That is,

$$\overrightarrow{\nabla}\left(\frac{f}{g}\right) = \frac{g\overrightarrow{\nabla}f - f\overrightarrow{\nabla}g}{g^2}.$$

Exercise 73. Let $f(x,y) = \dfrac{xy}{3xy+2}$. Determine $\overrightarrow{\nabla}f(1,-1)$.

Power Rule

Suppose that $f^{\alpha-1}$ and $\overrightarrow{\nabla} f$ exist. Then $\overrightarrow{\nabla} f^{\alpha}$ exists and

$$\overrightarrow{\nabla} f^{\alpha} = \alpha f^{\alpha-1} \overrightarrow{\nabla} f.$$

Exercise 74. Suppose $g(1,2,3) = 5$, and $\overrightarrow{\nabla} g(1,2,3) = (-1,2,0)$. Compute $\overrightarrow{\nabla} \sqrt[3]{g}$ at $(1,2,3)$.

Chain Rule

This one is a bit more involved. To begin, recall that if f is a real valued function of a single variable x and if x is a real valued function of a single variable t, then f can also be viewed as a function of x and

$$(f \circ x)'(t) = f'(x(t)) \cdot x'(t).$$

Let's consider now a function $f(x,y,z)$ and suppose that each of x, y, and z are functions of s and t. The we can view f as a function of s and t and it is possible to express $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ in terms of the partial derivatives of f with respect to x, y, and z and the partial derivatives of x, y, and z with respect to s and t. To wit:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

Exercise 75. Let f be a function of x, y, and z, and let each of x, y, and z be functions of s and t.

(a) Compute $\overrightarrow{\nabla} f$.

(b) Compute $\frac{\partial}{\partial s}(x,y,z) =$

(c) Compute $\overrightarrow{\nabla} f \cdot \frac{\partial}{\partial s}(x,y,z)$ and compare with the expression for $\frac{\partial f}{\partial s}$ above.

(d) Compute $\frac{\partial}{\partial t}(x, y, z) =$

(e) Compute $\vec{\nabla} f \cdot \frac{\partial}{\partial t}(x, y, z)$ and compare with the expression for $\frac{\partial f}{\partial t}$ above.

The work you did above suggest the general method for computing partial derivatives of a composition of multivariable functions.

Exercise 76. Let $g(x, y) = xy - 4$, $x(u, v) = 2uv + v$, and $y(u, v) = v^2 - u^2$.

(a) Compute $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$ using the chain rule.

(b) Compute $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$ by doing substitution first. (i.e. Plug $x = 2uv + v$ and $y = v^2 - u^2$ into g.)

(c) Compare the results from each computation.

7 Directional Derivatives and Level Sets

This worksheet discusses material corresponding roughly to sections 2.5 and 2.7 of your textbook.

Directional Derivatives

Let f be a real-valued function of n variables, $p \in \mathbb{R}^n$ a point, and $u \in \mathbb{R}^n$ a unit vector. Recall that the **directional derivative of f at p in the direction u** is

$$D_u f(p) = d_p f(u) = \vec{\nabla} f(p) \cdot u$$

and measures the rate of change in f as you move from the point p in the direction of u.

Exercise 77. Let $f(x, y) = x^3 + y^3 - x + y$. Suppose you are at the point $(-2, 3)$ in $\mathbb{R}^2$. If you move toward the point $(4, 1)$, what rate of change of f do you experience? (*Warning*: Both $(-2, 3)$ and $(4, 1)$ are *points* in this context.)

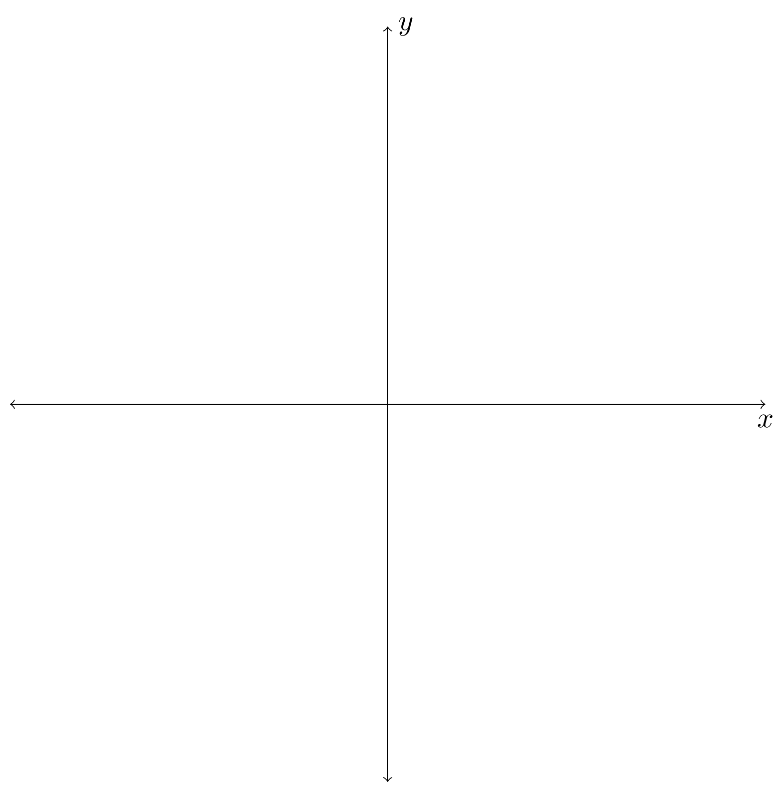
Exercise 78. Consider the same function $f(x, y) = x^3 + y^3 - x + y$. Let $i = (1, 0)$ and $j = (0, 1)$ be the standard basis vectors for $\mathbb{R}^2$. Compute $D_i f(x, y)$ and $D_j f(x, y)$. What do you notice?

Exercise 79. Consider the function $f(x, y, z) = xyz + x^2y - 3yz^2$. From the point $(-1, 2, -3)$, moving in what direction will result in the largest change in f? The smallest? (*Warning*: Directions are to be thought of as unit vectors.) Also determine the rate of change experienced in each of those directions.

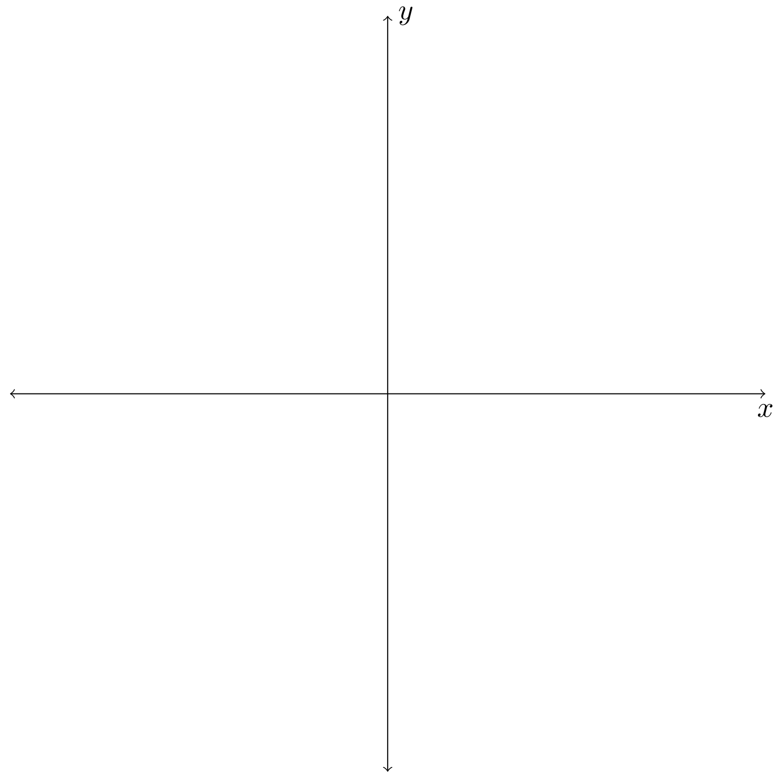
Level Sets

Suppose that $F(x_1, \ldots, x_n)$ is a real-valued function. The set of points in $\mathbb{R}^n$ which satisfy the equation $F(x_1, \ldots, x_n) = c$ is called the **level set where** $F = c$.

Exercise 80. Consider the function $F(x, y) = x^2 + y^2$. Draw the level sets where $F = c$ for c taking values of 4, 2, 1, 0, and -1.



Exercise 81. Consider the function $G(x, y) = x^2 - y^2$. Draw the level sets where $G = c$ for c taking values of 3, 2, 1, 0, -1, -2, and -3.



The level sets can be used to visualize the graph of a function. Of course, when graphing a function like $G(x, y) = x^2 - y^2$, we are thinking about the points $(x, y, z) \in \mathbb{R}^3$ where $z = x^2 - y^2$ and you can use some software to get a picture of the graph. Things get more difficult when we have more than two variables.

For example, if we have a function $F(x, y, z)$, then the graph of this function is the set of points $(x, y, z, w) \in \mathbb{R}^4$ which satisfy $w = F(x, y, z)$. These can be tricky to think about, but using the technique of level sets, we can start to understand what the graph "looks" like.

Exercise 82. Consider the function $F(x, y, z) = x^2 + y^2 + z^2$. Draw the level sets where $F = c$ for c taking values of 4, 2, 1, 0, and -1. Remember, the level sets will be collections of points in the domain of F, which in this case is $\mathbb{R}^3$.

Exercise 83. Consider surface in $\mathbb{R}^3$ determined by the equation $z^3 - x^3 + y^3 + xzy = 5$. Find a function $F(x, y, z)$ so that the given surface is a level set where $F = 0$.

Tangent Vectors

You've thought about tangent lines to curves and tangent planes to surfaces. We can also think about the tangent set to a point on the level set of a function. Here's the theorem that gets us working:

Theorem. Suppose F is continuously differentiable on a neighborhood of the point $p \in \mathbb{R}^{n+1}$ and that $\vec{\nabla} F(p) \neq 0$. Then, near p the level set M to F where $F = F(p)$ looks smooth and has dimension n. (I.e. The dimension of the level set is one less than the surrounding Euclidean space.)
In addition, a vector v is a **tangent vector to M at p** if

$$\vec{\nabla} F(p) \cdot v = 0.$$

Therefore, at p, there is a well defined tangent set to M which consists of all vectors $x \in \mathbb{R}^{n+1}$ such that $\vec{\nabla} F(p) \cdot (x - p) = 0$.

This is a lot to take in. Let's break it down in the case where our function F is a function of two variables.

Exercise 84. Consider $F(x, y) = x^2 + y^2$. Let $p = (-1, 2)$.

(a) Compute $F(p)$.

(b) Compute $\vec{\nabla} F(p)$.

(c) Determine an equation for the level set M to F where $F = F(p)$.

(d) Write down the equation $\vec{\nabla} F(p) \cdot (x+1, y-2) = 0$. This equation defines the tangent set to M at p.

(e) Plot both the level set M and the tangent set to M at p on the same axes.

(f) Observe that the tangent set to M at p is the tangent line to M at the point p.

(g) Use techniques from single variable calculus to compute the tangent line to M at p. That is, take the equation from part (c) and use implicit differentiation to find the slope of the tangent line, etc.

The same approach can be used to determine tangent planes to surfaces in $\mathbb{R}^3$ defined as level sets of some function $F(x, y, z)$.

Exercise 85. Let $F(x, y, z) = x^2 + y^2 + z^2$, and consider $p = (-1, 2, 1)$.

(a) Compute $F(p)$.

(b) Compute $\vec{\nabla} F(p)$.

(c) Determine an equation for the level set M to F where $F = F(p)$.

(d) Write down the equation $\vec{\nabla} F(p) \cdot (x + 1, y - 2, z - 1) = 0$. This equation defines the tangent set to M at p.

(e) Observe that the equation you obtained in the part (d) is the equation for a plane. This is the tangent plane to M at p.

Exercise 86. Determine if the following vectors are tangent to the given surface at the specified point.

(a) $v = (1, 2, 3)$, $11 = 3xyz - x^2y + z^3$, $p = (1, 5, 1)$.

(b) $w = (0, 1, -1)$, $\sin(xy) + \cos(yz) = -1$, $q = (1, \pi, -1)$.

8 Local Extrema

This worksheet discusses material corresponding roughly to sections 2.9 of your textbook.

Local Extrema

Recall that a **critical point** of a function $f = f(x, y)$ is a point p in the domain at which either f is not differentiable or $\vec{\nabla} f(p) = 0$. A critical point p is a **non-degenerate critical point** if $\vec{\nabla} f(p) = 0$ and $D = (f_{xx} f_{yy} - f_{xy}^2)(p) \neq 0$. We then have the second derivative test:

Theorem (Second Derivative Test)**.** Suppose that $f = f(x, y)$ has continuous second order partial derivatives near p and that p is a non-degenerate critical point of f. Then

1. If $D > 0$ and $f_{xx}(p) > 0$, then f attains a local minimum at p.
2. If $D > 0$ and $f_{xx}(p) < 0$, then f attains a local maximum at p.
3. If $D < 0$, then f has a saddle point at p.

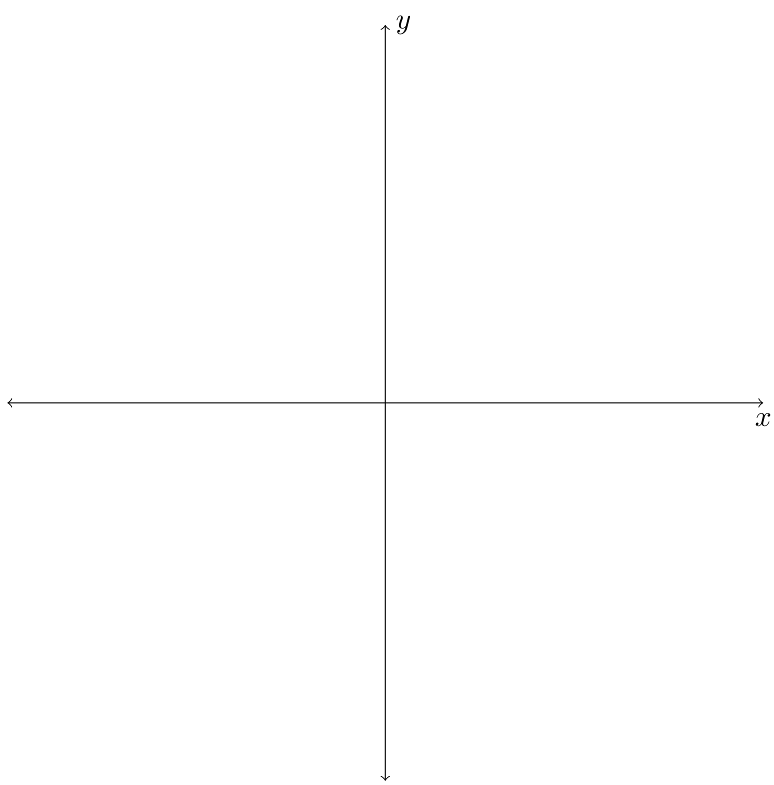
Exercise 87. Find and classify all non-degenerate critical points of the function $f(x, y) = -x^2 - y^2$.

Exercise 88. Find and classify all non-degenerate critical points of the function $g(x, y) = 3x - x^3 - 2y^2 + y^4$.

Exercise 89. Find and classify all non-degenerate critical points of the function $h(x, y) = x^3 + y^3 + 3x^2 - 3y^2$.

Exercise 90. Consider the function $f(x, y) = x^2 + y^2$.

(a) Sketch level curves for f where $f = 0$, 1, 4, and 9.



(b) For the level curve C where $f = 1$, compute the gradient vector for the following points on the curve: $(1, 0)$, $(\sqrt{2}/2, \sqrt{2}/2)$, $(0, 1)$, $(-\sqrt{2}/2, \sqrt{2}/2)$, $(-1, 0)$, $(-\sqrt{2}/2, -\sqrt{2}/2)$, $(0, -1)$, and $(\sqrt{2}/2, -\sqrt{2}/2)$.

(c) Sketch these gradient vectors on the curve C in your picture in part (a). (For this exercise, we're really interested in the direction of the vectors, so scale them so that they don't take up much space in the picture.)

(d) Do the same for the level curves where $f = 4$ and where $f = 9$. (That is, sketch the gradient vectors at the corresponding points to those curves.)

(e) Now, find the critical point for f.

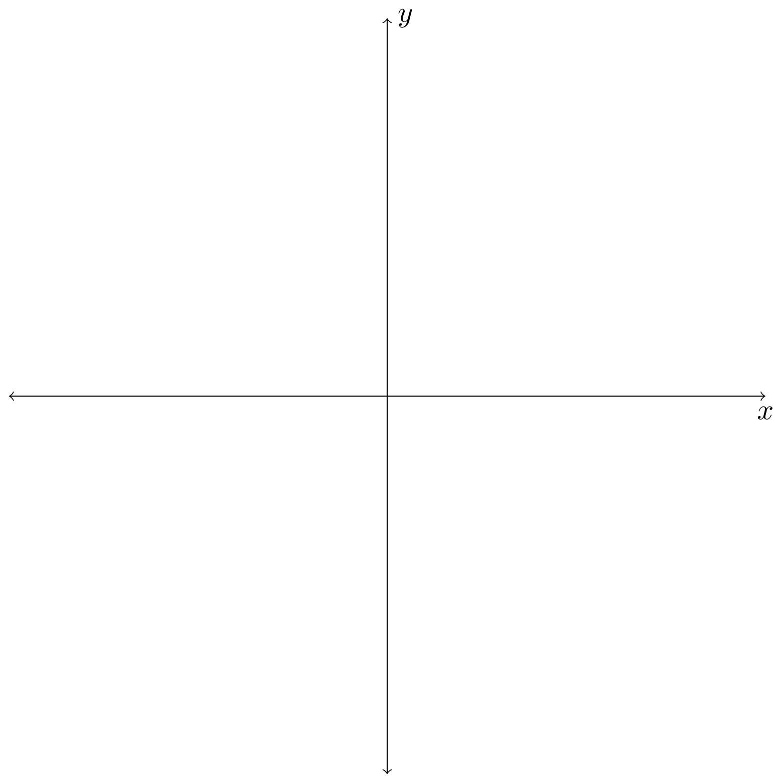
(f) Imagine you are standing at the critical point. Are the gradient vectors pointing toward you, or away from you?

(g) Still standing at the critical point, if you start to move in a direction, will the value of f increase or decrease? Or does it depend on what direction you go in?

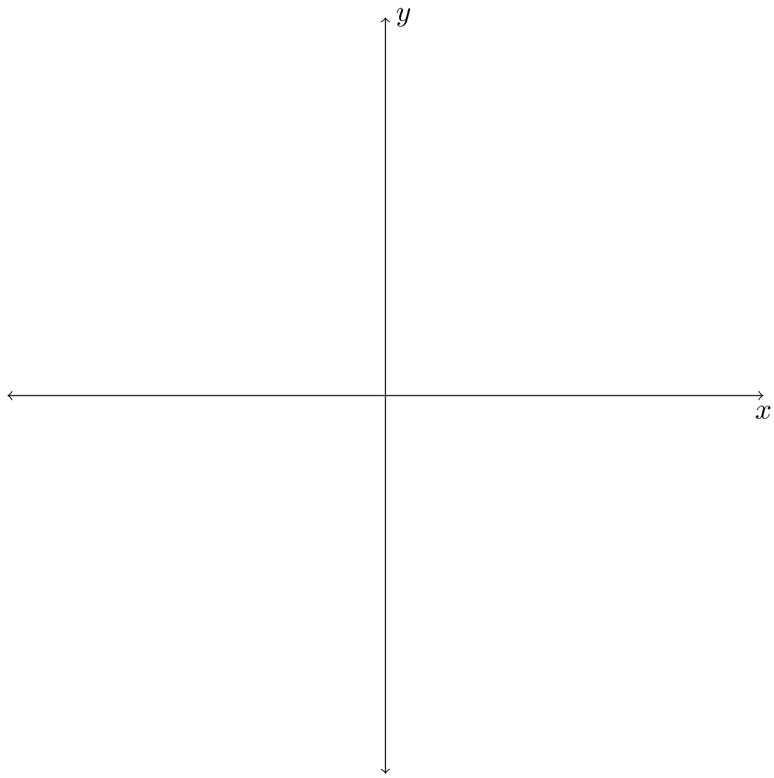
(h) From this analysis, is the critical point a local maximum, a local minimum, or a saddle point?

(i) Use the second derivative test to classify the critical point.

Exercise 91. Repeat the previous exercise for the function $g(x, y) = -x^2 - y^2$ and level curves where $g = 0$, -1, -4, and -9.



Exercise 92. Repeat the previous exercise for the function $h(x, y) = x^2 - y^2$ and level curves where $h = -9, -4, -1, 0, 1, 4,$ and 9.



9 Optimization

This worksheet discusses material corresponding roughly to sections 2.10 of your textbook.

Recall that to find global extrema for differentiable functions with compact domain, you look for critical points of the interior and those on the boundary.

Exercise 93. Consider the function $f(x, y) = xy$ defined on the closed rectangle $-1 \leq x \leq 2$, $-2 \leq y \leq 1$. Find the global extrema of f. (Notice, the boundary has four pieces to it which each need to be analyzed separately.)

Exercise 94. Consider the function $f(x, y) = xy$ defined on the disc $x^2 + y^2 \leq 1$.

(a) Find all critical points for f on the interior of the domain.

(b) Identify the boundary of the domain.

(c) Consider the function $p(t) = (\cos(t), \sin(t))$ with $0 \leq t \leq 2\pi$. Observe that the image of p is the boundary of the domain of f. This function p is a **parameterization** of the boundary.

(d) Using the parameterization p above, restrict the function f to the boundary of its domain. Express f with this restriction as a function of t.

(e) Using the restricted function f, find all critical points of f restricted to the boundary.

(f) List all critical points in the interior of f, on the boundary of f, and also the point $(1, 0)$. (Thing: Why are we also considering the point $(1, 0)$?)

(g) Determine the points where f achieves its global maximum and its global minimum. Also determine the global maximum value and the global minimum value for f.

Exercise 95. Consider the function $T(x, y, z) = x^2 + y^2 - z^2 - x - y$ defined on the solid cylinder in $\mathbb{R}^3$ given by $-1 \leq z \leq 0$ and $x^2 + y^2 \leq 1$.

(a) Sketch the domain C of T.

(b) Describe the boundary of C.

(c) Find all critical points of T which lie in the interior of C.

(d) Find all critical points of T which lie on the interior of the "top" of the cylinder.

(e) Find all critical points of T which lie on the boundary of the "top" of the cylinder.

(f) Find all critical points of T which lie on the interior of the "bottom" of the cylinder.

(g) Find all critical points of T which lie on the boundary of the "bottom" of the cylinder.

(h) Find all critical points of T which lie on the interior of the "side" of the cylinder by first finding a parameterization for the side of the cylinder. (*Hint*: The x and y values for points on the side of the cylinder all lie on a circle of radius 1, which you can parameterize using cosine and sine.)

(i) Observe that the boundary of the side is the boundaries of the top and bottom. Hence, you've already found critical points there.

(j) Determine the points at which T attains its global extrema. Also determine the global extreme values for T.

10 Partial Anti-Differentiation and Iterated Integrals

This worksheet covers material from section 3.1 of your text.

Partial Anti-Derivatives

First, the single variable calculus case.

Exercise 96. Suppose $f(x)$ is a real-valued function of a single real variable. Also, suppose you know that $\frac{df}{dx} = 3x^2 - 4x + 2$. What is $f(x)$?

Now, the multi-variable case.

Exercise 97. Compute $\frac{\partial f}{\partial x}$ for each of the functions below.

(a) $f(x, y) = x^3 - 2x^2y + 2xy$

(b) $f(x, y) = x^3 - 2x^2y + 2xy + 5$

(c) $f(x, y) = x^3 - 2x^2y + 2xy + y^7$

(d) $f(x, y) = x^3 - 2x^2y + 2xy + \sin(y) + e^{\sqrt{y}}$

(e) $f(x, y) = x^2 - 2x^2y + 2xy + y^{10}\ln|\sin(y) + \cos(y)| + \dfrac{y^2}{10y^7 - y^6 + 1}$

Exercise 98. Suppose $f(x, y)$ is a real-valued function of two real variables, and suppose $\frac{\partial f}{\partial x} = 3x^2 - 4xy + 2y$. What is $f(x, y)$?

Exercise 99. Suppose $g(x, y)$ is a real-valued function of two real variables, and suppose $\frac{\partial g}{\partial y} = 3x^2 - 4xy + 2y$. What is $g(x, y)$?

Iterated Integrals

Exercise 100. Evaluate the definite integral below.

$$\int_1^2 3x^2 - 4xy + 2y \, dx$$

Exercise 101. Evaluate the definite integral below.

$$\int_0^1 3x^2 - 4xy + 2y \, dy$$

Exercise 102. Evaluate the definite integral below.

$$\int_y^{y^2+1} 3x^2 - 4xy + 2y \, dx$$

Exercise 103. Evaluate the **iterated integral** below.

$$\int_1^2 \left[\int_y^{y^2} 3x - 2y \, dx \right] dy$$

The brackets in the above notation make it clear what order to integrate in. We seldom use these brackets, however, and the iterated integral above is typically written

$$\int_1^2 \int_y^{y^2} 3x - 2y \, dx \, dy$$

and it is implied that you start on the inside and work your way out.

Exercise 104. Compute:

$$\int_{\pi/4}^{\pi/2} \int_0^{\sin(x)} \frac{1+2y}{\sin(x)} \, dy \, dx$$

Of course, all of this can be done with even more variables.

Exercise 105. Let $f(x, y, z)$ be a real valued function, and suppose $\dfrac{\partial f}{\partial x} = xyz + 3xy + y^2x + 4$. Determine $f(x, y, z)$.

Exercise 106. Evaluate:

$$\int_0^2 \int_z^1 \int_z^{yz} xyz \; dx \; dy \; dz$$

11 Polar Coordinates

This worksheet covers material from section 3.3 of your text.

Recall that a point in $\mathbb{R}^2$ with Cartesian coordinates (x, y) can be described with polar coordinates (r, θ) where

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

$$x^2 + y^2 = r^2$$

$$\theta = \begin{cases} \arctan(y/x) & \text{if } x > 0 \\ \arctan(y/x) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \pi/2 & \text{if } x = 0 \text{ and } y > 0 \\ -\pi/2 & \text{if } x = 0 \text{ and } y < 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

Exercise 107. Convert the Cartesian coordinates below to polar coordinates.

(a) $(0, 4)$

(b) $(1, 1)$

(c) $(-2, \sqrt{2})$

(d) $(-4, -4)$

(e) $(0, -3)$

(f) $(2, -2)$

Exercise 108. Convert the polar coordinates below to Cartesian coordinates.

(a) $(1, 0)$

(b) $(3, \pi/4)$

(c) $(6, \pi/2)$

(d) $(2, 5\pi/6)$

(e) $(0, \pi)$

(f) $(3, 7\pi/4)$

Exercise 109. Sketch the regions in the plane described by the inequalities below.

(a) $r \leq 3$

(b) $r \geq 2$ and $\pi \leq \theta \leq 3\pi/2$

(c) $1 \leq r \leq 2$

(d) $0 \leq \theta \leq \pi/4$ and $0 \leq r \leq \sec(\theta)$.

Exercise 110. The change of coordinates from polar to Cartesian can be viewed as a function $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(r,\theta) = (r\cos\theta, r\sin(\theta)) = (x,y)$$

(a) Compute:

$$[dF] = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}$$

(b) Compute the determinant $\det[dF]$.

(c) Write $dA = |\det[dF]|\ dr\ d\theta$.

(d) Observe that you have $dA = r\ dr\ d\theta$. Hence, if we integrate using polar coordinates, the area element $dy\ dx$ is replaced by $r\ dr\ d\theta$.

Exercise 111. Evaluate the integral below, where R is the portion of the unit disc in the first quadrant.

$$\iint_R x^2 + y^2 \, dA$$

Exercise 112. Evaluate the integral below, where R is the region in the third quadrant between the circles centered at the origin of radii 1 and 2.

$$\iint_R y \, dA$$

Exercise 113. Evaluate the integral below, where R is the region bounded by the lines $y = x$, $y = 0$, $x = 1$, and $x = 2$.

$$\iint_R \frac{1}{\sqrt{x^2 + y^2}} \, dA$$

12 Integration in $\mathbb{R}^3$

This worksheet covers material from section 3.4 of your text.

First, some review.

Exercise 114. Consider the region R in $\mathbb{R}^2$ which is the rectangle given by $-2 \leq x \leq 3$, $0 \leq y \leq 2$.

(a) Sketch the region R.

(b) Determine the integral $\iint_R f(x,y)\, dA$ where $f(x,y) = 2x^2y$.

Exercise 115. Consider the region R in $\mathbb{R}^2$ which is the region in the first quadrant bounded by $y = \sqrt{2 - x^2}$ and the coordinate axes.

(a) Sketch the region R.

(b) Determine the integral $\iint_R f(x, y)\, dA$ where $f(x, y) = 2x^2y$.

Exercise 116. Consider the region R in $\mathbb{R}^3$ which is the rectangular box given by $-1 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 1$.

(a) Sketch the region R.

(b) Determine the integral $\iiint_R f(x, y, z)\, dV$ where $f(x, y, z) = 2xy + 3yz$.

Exercise 117. Consider the region R in $\mathbb{R}^3$ in the first octant (i.e. where x, y, and z are all non-negative) between the xy-plane and the curve $z = 2 - x^2 - y^2$.

(a) Sketch the region R.

(b) $\iiint_R f(x, y, z)\, dV$ where $f(x, y, z) = y$. Use $dV = dzdydx$.

(c) $\iiint_R f(x,y,z)\,dV$ where $f(x,y,z) = y$. Use $dV = dydzdx$.

(d) $\iiint_R f(x,y,z)\,dV$ where $f(x,y,z) = y$. Use $dV = dxdydz$.

(e) $\iiint_R f(x,y,z)\, dV$ where $f(x,y,z) = y$. Use $dV = dzdxdy$.

(f) $\iiint_R f(x,y,z)\, dV$ where $f(x,y,z) = y$. Use $dV = dydxdz$.

(g) $\iiint_R f(x,y,z)\,dV$ where $f(x,y,z)=y$. Use $dV = dydzdx$.

(h) $\iiint_R f(x,y,z)\,dV$ where $f(x,y,z)=y$. Use $dV = dxdzdy$.

13 Cylindrical Coordinates

This worksheet covers material from section 3.6 of your text.

Recall that in $\mathbb{R}^3$, a point with Cartesian coordinates (x, y, z) has cylindrical coordinates (r, θ, z) where $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$.

Exercise 118. Sketch the graph of the following surfaces in $\mathbb{R}^3$ with the given equations in cylindrical coordinates.

(a) $z = 2$

(b) $r = 1$

(c) $\theta = \pi/4$

(d) $z = r$

(e) $r = 1$ and $z = \theta$

Exercise 119. Use cylindrical coordinates to compute the volume of a cylinder with base radius a and height b.

Exercise 120. Sketch the region S which is bounded by the cone $z = \sqrt{x^2 + y^2}$ and the hemisphere $z = \sqrt{18 - x^2 - y^2}$. Use cylindrical coordinates to compute the volume of the region S.

Exercise 121. Consider the cube C in $\mathbb{R}^3$ given by $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$.

(a) Describe the plane determined by each face of the cube using cylindrical coordinates.

(b) Use cylindrical coordinates to compute the volume of C.

14 Practice

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For each exercise below, sketch the region described. Set up the described integral in each of Cartesian, cylindrical, and spherical coordinates. You do not need to evaluate the integral.

1. The region S is the unit cube in $\mathbb{R}^3$ where $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$. The integral is $\iiint_S f(x, y, z)\, dV$ where $f(x, y, z) = 2x + y - xz$.

2. The region S is the region in $\mathbb{R}^3$ between the cylinder with base on the xy-plane of radius 4 and height 6 and the sphere centered at the origin of radius 4 . The integral is $\iiint_S f(x, y, z)\, dV$ where $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$.

3. The region S is the region inside the right circular cylinder given by $x^2 + y^2 = 9$, above the plane where $z = 0$, and below the plane where $z = y$. The integral is $\iiint_S f(x, y, z)\, dV$ where $f(x, y, z) = 1$.

15 Average Value, Mass

Average Value

Recall that the average value of a function f on a region B is $\frac{1}{\text{length}(B)} \int_B f\,dx$ if B is a region in $\mathbb{R}$, $\frac{1}{\text{area}(B)} \iint_B f\,dA$ if B is a region in $\mathbb{R}^2$, and $\frac{1}{\text{volume}(B)} \iiint_B f\,dV$ if B is a region in $\mathbb{R}^3$.

Exercise 122. Find the average value of the function $f(x, y) = x^2 - xy$ on the region bounded by the line $y = 0$ and the semicircle $y = \sqrt{1 - x^2}$.

Exercise 123. Find the average value of the function $f(x, y, z) = \frac{1}{x^2 + y^2 + z^2}$ on the region between the spheres $x^2 + y^2 + z^2 = 3$ and $x^2 + y^2 + z^2 = 4$.

Density and Mass

You may recall from a physics class that density is mass per volume. That is $\delta = \dfrac{m}{V}$.

Exercise 124. Solve this equation for m.

So, $m = \delta V$. If we have an object which occupies a region S in $\mathbb{R}^3$, then the density may vary with the point in the object. So to find the mass of the object, we'd want to integrate dm. That is

$$m = \iiint_S \, dm$$

Since $m = \delta V$, then $dm = \delta \, dV$. Hence,

$$m = \iiint_S \delta(x, y, z) \, dV$$

Exercise 125. The solid tetrahedron in $\mathbb{R}^3$ with vertices at $(0 \text{ cm}, 0 \text{ cm}, 0 \text{ cm})$, $(1 \text{ cm}, 0 \text{ cm}, 0 \text{ cm})$, $(0 \text{ cm}, 1 \text{ cm}, 0 \text{ cm})$, and $(0 \text{ cm}, 0 \text{ cm}, 1 \text{ cm})$ has density given by $\delta(x, y, z) = x + yz + 2 \text{ g/cm}^3$. Determine the mass of the tetrahedron.

Exercise 126. The solid ball $x^2+y^2+z^2 \leq 4$ cm in $\mathbb{R}^3$ has density given by $\delta(x, y, z) = 2\sqrt{x^2 + y^2 + z^2}$ g/cm^3. Determine the mass of the ball.

The same considerations can be made for two dimensional regions R. We can consider the area density of an object to be $\delta = m/A$. Hence, $m = \delta A$, so $dm = \delta \, dA$, and

$$m = \iint_R \delta(x, y, z) \, dA$$

Exercise 127. Let R be the region in R^2 between the line $y = 3$ and the curve $y = x^2$, where both x and y are measured in cm. Suppose this region has area density given by $\delta(x, y) = xy + 4$ g/cm^2. Determine the mass of the region.

Exercise 128. Let R be the unbounded region in $\mathbb{R}^2$ between the lines $x = 1$ and $y = 0$ and the curve $xy = 1$. Suppose this region has density given by $\delta(x, y) = y/x$ g/cm^2. Determine the mass of the region.

16 Centroid and Center of Mass

Recall that the centroid of a region S in $\mathbb{R}^3$ is the point $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{1}{V} \iiint_S x \, dV$$

$$\bar{y} = \frac{1}{V} \iiint_S y \, dV$$

$$\bar{z} = \frac{1}{V} \iiint_S z \, dV$$

$$V = \iiint_S \, dV$$

If an object which occupies the region S has a density function $\delta(x, y, z)$, then the center of mass of the object is the point $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{1}{M} \iiint_S x \, dm = \frac{1}{M} \iiint_S x\delta(x, y, z) \, dV$$

$$\bar{y} = \frac{1}{M} \iiint_S y \, dm = \frac{1}{M} \iiint_S y\delta(x, y, z) \, dV$$

$$\bar{z} = \frac{1}{M} \iiint_S z \, dm = \frac{1}{M} \iiint_S z\delta(x, y, z) \, dV$$

$$M = \iiint_S \, dm = \iiint_S \delta(x, y, z) \, dV$$

Analogous statements hold for regions in $\mathbb{R}^2$.

Exercise 129. Determine the centroid, mass, and center of mass of the cube $1 \le x \le 3$, $1 \le y \le 3$, $1 \le z \le 3$ with density function $\delta(x, y, z) = 18x + 36y + 12z$.

Exercise 130. Determine the centroid, mass, and center of mass of the solid region between the spheres centered at the origin of radii 4 and 6, with density function $\delta(x, y, z) = 4/(x^2 + y^2 + z^2)$.

Exercise 131. Determine the centroid, mass, and center of mass of the solid cone $z = \sqrt{x^2 + y^2}$, $z \leq 3$ with density function $\delta(x, y, z) = x$.