MATH141B
Calculus & Analytic Geometry II
Workbook

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Contents

Introduction

This document is a compilation of worksheets I have created for use in a first semester calculus course. My course met five days a week for 50 minutes per class. The weekly routine was (roughly) worksheet, lecture, computer lab, worksheet, lecture. That is, the worksheets were separated by class periods during which I lectured. Below, if a worksheet seems to be assuming knowledge of content which has not been covered by previous worksheets, it is likely that this content was covered in lecture. Also, each worksheet reference sections of a textbook. For my class, we used "Worldwide Integral Calculus with infinite series" by David B. Massey and published by Worldwide Center of Mathematics, and all section references below are to this textbook. Of course, these worksheets can be useful in this study of calculus in the absence of this, or any, textbook.

If you notice any typos or other errors, or are interested in my LaTeX code, please send me an email: wkronholm@whittier.edu. I would also appreciate hearing from instructors who use these materials in their courses.

Chapter 1

Anti-derivatives

1.1 Anti-derivatives

This worksheet discusses material from section 1.1 of your book.

We begin with some friendly review.

Exercise 1.1.1. Compute derivatives of each of the functions below.

(a) $f(x) = \sin(3x)$

(b) $g(t) = t\exp(t)$

(c) $h(s) = \dfrac{\ln|s|}{s}$

(d) $i(z) = \dfrac{z^4 - z^2 + 1}{(z+2)^3}$

Exercise 1.1.2. Determine all points at which the function below is **not** differentiable.

$$f(x) = \begin{cases} \exp(-x) - 1 & \text{if } x < -2 \\ x & \text{if } -2 \leq x \leq 2 \\ x^2 - 3x + 4 & \text{if } x \geq 2 \end{cases}$$

Find a description for f'.

A solution $y = F(x)$ to the equation $\frac{dy}{dx} = f(x)$ is called an **anti-derivative** of f.

Exercise 1.1.3. Find at least three solutions to each of the equations below.

(a) $\frac{dy}{dx} = 2x$

(b) $\frac{dy}{dx} = \exp(x)$

(c) $\frac{dy}{dx} = 3x^2 - \sin(x) + \frac{1}{x} + 5$

(d) $\frac{dy}{dx} = \frac{1}{1 + x^2}$

(e) $\frac{dy}{dx} = \exp(-x) + 1$

As you saw above, anti-derivatives are not unique. In fact, if $F(x)$ is an anti-derivative of $f(x)$, then so is $F(x) + C$ for any constant C. In this case, we call $F(x) + C$ the **general anti-derivative** of f. The notation for the general anti-derivative of f is

$$\int f(x)\, dx = F(x) + C$$

Exercise 1.1.4. Determine the following general anti-derivatives.

(a) $\int 0\, dx$

(b) $\int 1\, dx$

(c) $\int x^p\, dx$, if $p \neq -1$.

(d) $\int x^{-1}\, dx$

(e) $\int \exp(x)\, dx$

(f) $\int \sin(x)\, dx$

(g) $\int \cos(x)\, dx$

(h) $\int \frac{1}{1+x^2}\, dx$

(i) $\int \frac{1}{\sqrt{1-x^2}}\, dx$

(j) $\int \sec(x)\tan(x)\, dx$

Exercise 1.1.5. Determine $\int |x| \, dx$. It may help to recall that $|x|$ can be considered as a piecewise defined function:

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

1.2 Integration by Parts

This worksheet discusses material corresponding roughly to section 1.1 of your textbook.

Recall the product rule for differentiation. Suppose u and v are functions of x. Then

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

Exercise 1.2.1. Rewrite the product rule for differentiation as a rule for anti-derivatives. You should have two $\int$'s on one side of the equation.

Exercise 1.2.2. Solve your previous equation for $\int u\frac{dv}{dx}\,dx$.

Exercise 1.2.3. Check that your equation is equivalent to the one below.

$$\int u\,dv = uv - \int v\,du$$

The equation $\int u\,dv = uv - \int v\,du$ is referred to as **integration by parts**. It gives us a technique for finding anti-derivatives as shown in the example below.

Exercise 1.2.4. Consider the anti-derivative $\int x\exp(x)\,dx$.

(a) Let $u = x$ and $dv = \exp(x)dx$. Observe that the original anti-derivative is of the form $\int u\,dv$.

(b) Determine du and v.

(c) Apply integration by parts. That is, plug u, v, du and dv into the equation $\int u\,dv = uv - \int v\,du$.

(d) Compute the remaining anti-derivative.

Exercise 1.2.5. Use integration by parts to determine the following anti-derivatives.

(a) $\int x \cos(x)\, dx$

(b) $\int 3x^2 \exp(x)\, dx.$

(c) $\int 2x \exp(-2x)\, dx$

(d) $\int \sqrt{x} \ln(x)\, dx$

(e) $\int \exp(x) \sin(x)\, dx$

(f) $\int \ln(x)\, dx$

(g) $\int x \tan^2(x)\, dx$

(h) $\int \arctan(x)\, dx$

1.3 A Note on Domains

We're used to seeing statements like $\int f(x)\,dx = F(x) + C$, where C is a constant. We really need to be careful though when working with functions with disconnected domains.

Exercise 1.3.1. Consider the function $f(x) = \dfrac{1}{x^2}$.

(a) What is the domain of f?

(b) Sketch a graph of f.

(c) Check that $F(x) = -\dfrac{1}{x}$ is an anti-derivative for f. (i.e. Compute $F'(x)$.)

(d) Let G be the piecewise defined function below.

$$G(x) = \begin{cases} -\frac{1}{x} + 2 & \text{if } x < 0 \\ -\frac{1}{x} - 3 & \text{if } x > 0 \end{cases}$$

Check that G is also an anti-derivative for f.

(e) Is $G(x) = F(x) + C$? That is, do the two anti-derivatives differ by a constant? Explain.

The issue with the previous example is that the domain of f is disconnected. That is, it is comprised of two separate pieces: $(\infty, 0)$ and $(0, \infty)$. When the domain of our function is disconnected, then the "constant" of integration can be different on each connected component of the domain. So the general anti-derivative needs to take this into account.

Exercise 1.3.2. Again, let $f(x) = \dfrac{1}{x^2}$.

(a) Check that F given below is the general anti-derivative of f.

$$F(x) = \begin{cases} -\frac{1}{x} + C_1 & \text{if } x < 0 \\ -\frac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

(b) Sketch a graph of F.

Exercise 1.3.3. Compute the general anti-derivative $\int \frac{1}{x}\, dx$ taking into account the disconnected domain of the integrand.

Exercise 1.3.4. Compute the general anti-derivative $\int \frac{3}{2x-3}\, dx$ taking into account the disconnected domain of the integrand.

Exercise 1.3.5. Compute the general anti-derivative $\int \sec^2(\pi\theta)\,d\theta$ taking into account the disconnected domain of the integrand.

Exercise 1.3.6. Compute the general anti-derivative $\int \sec(\pi\theta)\,d\theta$ taking into account the disconnected domain of the integrand.

1.4 Partial Fraction Decomposition

Below, we will work on a technique which will allow us to find anti-derivatives for rational functions. The technique is completely algebraic and is inspired by our ability to compute the anti-derivatives below with ease.

Exercise 1.4.1. Compute the following anti-derivatives.

(a) $\displaystyle\int \frac{1}{x}\,dx$

(b) $\displaystyle\int \frac{1}{x^2}\,dx$

(c) $\displaystyle\int \frac{1}{x^3}\,dx$

(d) $\displaystyle\int \frac{1}{x^2+1}\,dx$

(e) $\displaystyle\int \frac{1}{2x-3}\,dx$

(f) $\displaystyle\int \frac{1}{(2x-3)^2}\,dx$

(g) $\displaystyle\int \frac{1}{2x^2+3}\,dx$ (*Hint*: Factor out a scalar.)

(h) $\displaystyle\int \frac{5x}{2x^2+3}\,dx$ (*Hint*: This one is easier than the last one.)

The idea is to take a given rational function and decompose it into pieces like the ones in the previous exercise. The general process is outlined in the exercise below.

Exercise 1.4.2. Let $f(x) = \dfrac{2x+1}{x^2-3x+2}$.

(a) Factor the denominator.

(b) What is the degree of each factor of the denominator? That is, are they degree 1 polynomials? Degree 2? Degree 14?

(c) For each factor, write a new rational function in the form $\dfrac{p(x)}{\text{FACTOR}}$ where $p(x)$ is a polynomial of one degree less than the degree of the factor. It should be a polynomial with unknown coefficients, so it p should be degree 1, write $Ax + B$. If it's degree 4, write $Ax^4 + Bx^3 + Cx^2 + Dx + E$, etc. Use new labels for the coefficients for each of these new rational functions.

(d) Now that you have the two pieces you need, write $\dfrac{2x+1}{x^2-3x+2} = \dfrac{A}{\text{FACTOR\#1}} + \dfrac{B}{\text{FACTOR\#2}}$.

(e) Multiply the denominator on the left to the right, and cancel factors on the right.

(f) Distribute on the right to get something of the form (SOME STUFF)x + (OTHER STUFF).

(g) Equate coefficients, and write down the system of equations. You should have two equations in the two unknowns A & B.

(h) Solve the system of equations.

(i) Now, use your work to compute $\int \frac{2x+1}{x^2-3x+2}\,dx$.

Exercise 1.4.3. Compute $\int \frac{1}{(x+1)(x-2)(x+3)}\,dx$. Follow the method in the previous exercise, but use three terms instead of two.

Exercise 1.4.4. Consider the function $f(x) = \dfrac{2x+3}{(x-1)^2}$.

(a) Find A and B so that $2x + 3 = A(x - 1) + B$.

(b) Use your A and B to write $\dfrac{2x+3}{(x-1)^2} = \dfrac{A(x-1)}{(x-1)^2} + \dfrac{B}{(x-1)^2}$. Simplify.

(c) Compute $\displaystyle\int \frac{2x+3}{(x-1)^2}\,dx$.

Exercise 1.4.5. Compute $\int \frac{1}{(x^2+1)(x-3)^2}\, dx$.

Chapter 2

Sequences and Series

2.1 Sequences

A sequence is a function $a\colon \mathbb{Z}_{\geq m} \to \mathbb{R}$. That is, it is a function with domain the integers greater than some given integer m (usually, $m = 0$ or $m = 1$) with codomain the real numbers. Perhaps more intuitively, a sequence is a list of numbers. Instead of the usual functional notation $a(n)$, we tend to write a_n for the value of the function a at the input n.

For example, if $a_n = 3n - 4$ for $n \geq 4$, then the sequence is the following:

$$8, 11, 14, 17, 20, \ldots$$

Exercise 2.1.1. List the first five terms of the sequences below.

(a) $a_n = n$, $n \geq 0$.

(b) $b_n = -2n - 1$, $n \geq 2$.

(c) $c_n = \dfrac{2n+1}{n}$, $n \geq 1$.

(d) $d_n = \left(\dfrac{2}{3}\right)^n$, $n \geq 0$.

(e) $e_n = (-1)^n$, $n \geq 0$.

Exercise 2.1.2. Determine a closed form expression for the sequences below. That is, write each as $a_n =$ something, $n \geq m$. (You get to choose m.) There are many (many!) correct expressions.

(a) $2, 5, 8, 11, 14, \ldots$

(b) $-3, 6, -12, 24, -48, \ldots$

(c) $\dfrac{2}{3}, \dfrac{4}{5}, \dfrac{6}{7}, \dfrac{8}{9}, \dfrac{10}{11}, \ldots$

(d) $-1, 1, -1, 1, -1, \ldots$

(e) $1, 2, 1, 4, 1, 6, 1, 8, \ldots$

For us, the most important feature of a sequence will be whether or not it converges, and, in the case of convergence, what it converges to. The definition of convergence should look familiar:

A sequence a_n **converges** to the real number L if given and $\varepsilon > 0$ there is an integer N so that whenever $n \geq N$, $|a_n - L| < \varepsilon$. That is, if a_n is always close to L when n is big, then a_n converges to L. We write this in the usual way: $\lim_{n\to\infty} a_n = L$. Otherwise, the sequence **diverges**.

If a_n is a sequence and for any $M > 0$ there is an integer N so that whenever $n \geq N$, $a_n > M$, then we say a_n **diverges to infinity** and write $\lim_{n\to\infty} a_n = \infty$. If a_n is a sequence and for any $M < 0$ there is an integer N so that whenever $n \geq N$, $a_n < M$, then we say a_n **diverges to negative infinity** and write $\lim_{n\to\infty} a_n = -\infty$.

For example, $\lim_{n\to\infty} 3n - 4 = \infty$ and $\lim_{n\to\infty} \dfrac{3n-4}{2n+1} = \dfrac{3}{2}$.

Exercise 2.1.3. For each of the sequences on the previous page, determine which converge and which do not. In case of convergence, determine the limit. In case of divergence, determine which diverge to infinity, which diverge to negative infinity, and which simply diverge.

Exercise 2.1.4. Compute the limit as n goes to ∞ of the sequences below. Most of the usual rules for computing limits of functions apply in the context of sequences.

(a) $a_n = \dfrac{1}{n}$, $n \geq 1$

(b) $b_n = \dfrac{3n-4}{1-4n}$, $n \geq 0$

(c) $c_n = \left(\dfrac{3}{4}\right)^n$, $n \geq 0$

(d) $d_n = \dfrac{1}{n} - \dfrac{1}{n-1}$

(e) $e_n = \sin(\pi n)$

Exercise 2.1.5. Give an example of a sequence a_n and a continuous, non-zero function f so that the sequence a_n diverges, but the sequence $f(a_n)$ converges. Show that both a_n and $f(a_n)$ have the stated properties.

Exercise 2.1.6. Create a sequence a_n so that $0 \leq a_n \leq 3$ for all n, and a_n is **increasing** in the sense that $a_n \leq a_{n+1}$ for all n. Show that your sequence has the required properties. Does your sequence converge?

Exercise 2.1.7. Create a sequence a_n so that $1 \leq a_n \leq 4$ for all n, and a_n is **decreasing** in the sense that $a_n \geq a_{n+1}$ for all n. Show that your sequence has the required properties. Does your sequence converge?

Exercise 2.1.8. Create a sequence a_n so that $2 \leq a_n \leq 3$ for all n, and a_n diverges. Is a_n increasing? Decreasing?

Exercise 2.1.9. Is it true that if $a_n \leq b_n$ for all n, and b_n is convergent, then a_n is also convergent? If so, explain why. If not, give an example to show this.

Exercise 2.1.10. Is it true that if $a_n \geq b_n$ for all n, and b_n is divergent, then a_n is also divergent? If so, explain why. If not, give an example to show this.

2.2 Summations

Given a sequence a_k, the **summation** of a_k as k goes from m to n is denoted

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

For example, $\sum_{k=1}^{5} k^2 = 1 + 4 + 9 + 16 + 25$.

Exercise 2.2.1. Determine the summations below.

(a) $\sum_{k=-2}^{5} k$

(b) $\sum_{j=0}^{3} 2j^2 + 5j - 1$

(c) $\sum_{k=0}^{5} 2$

(d) $\sum_{\ell=-3}^{3} \ell^3$

Exercise 2.2.2. Combine the summations below into a single summation. Do not compute the sum.

$$\sum_{k=0}^{25}(k+1)(k-1)+3\sum_{k=0}^{25}k(k+1)$$

Exercise 2.2.3. Combine the summations below into a single summation. Do not compute the sum.

$$\sum_{k=0}^{25}(k+1)(k-1)+3\sum_{j=0}^{25}j(j+1)$$

Exercise 2.2.4. Combine the summations below into a single summation. Do not compute the sum.

$$\sum_{k=0}^{25}(k+1)(k-1)+3\sum_{k=2}^{25}k(k+1)$$

Given a function f, the **finite difference** function Δf is given by $(\Delta f)(k) = f(k) - f(k-1)$. For example, if $f(x) = 2x + 1$, then $(\Delta f)(3) = f(3) - f(2) = 7 - 5$.

Exercise 2.2.5. Let $f(x) = x^2 - x + 1$. Compute the following.

(a) $\Delta f(1)$

(b) $\Delta f(2)$

(c) $\Delta f(3)$

(d) $\Delta f(4)$

(e) $\sum_{k=1}^{4} \Delta f(k)$

Exercise 2.2.6. Compute the following.

(a) Δk

(b) Δk^2

(c) Δk^3

(d) $\Delta 1$

Exercise 2.2.7. Let f be a function.

(a) Expand the summation $\sum_{k=1}^{3} \Delta f(k)$.

(b) Is there any simplification that occurs? If so, simplify.

(c) Without expanding, simplify: $\sum_{k=1}^{10} \Delta f(k)$.

(d) Without expanding, simplify: $\sum_{k=3}^{10} \Delta f(k)$.

(e) Without expanding, simplify: $\sum_{k=m+1}^{n} \Delta f(k)$.

Summations of the form $\sum_{k=m+1}^{n} \Delta f(k)$ are called **telescoping sums**, a throwback to the collapsible telescopes of yore. These are some of the easiest summations to deal with, since we can determine the value of the sum computing all of the terms of the sequence being summed.

Exercise 2.2.8. (a) Show that $k = \Delta\left[\frac{k(k+1)}{2}\right]$. That is, show that the right-hand side of the equation simplifies to k.

(b) Use the previous part to determine $\sum_{k=1}^{n} k$.

(c) Show that $k^2 = \Delta\left[\frac{k(k+1)(2k+1)}{6}\right]$.

(d) Determine $\sum_{k=1}^{n} k^2$.

Recall that a sequence of the form $a_n = b \cdot r^n$ where b and r are fixed real numbers is called a **geometric** sequence.

Exercise 2.2.9. (a) Show that $r^k = \Delta \left[\dfrac{r^{k+1}}{r-1} \right]$, provided $r \neq 1$.

(b) If $r \neq 1$, compute $\displaystyle\sum_{k=0}^{n} br^k$.

(c) Compute $\displaystyle\sum_{k=0}^{10} 3 \left(\frac{4}{5} \right)^k$.

(d) Compute $\displaystyle\sum_{k=0}^{8} \left(-\frac{5}{4} \right)^k$.

2.3 Geometric Series

Recall that a geometric sequence is a sequence of the form $a_n = b \cdot r^n$, $n \geq 0$. A **geometric series** therefore is a series of the form $\sum_{k=0}^{\infty} br^k$.

To determine the limit of a geometric series, we first will compute the partial sums.

Exercise 2.3.1. Consider the series $\sum_{k=0}^{\infty} r^n$ where r is a fixed real number. Let $s_n = \sum_{k=0}^{n} r^k$ be the nth partial sum.

(a) Expand the sum for s_n. That is, write out all of the terms in the summation. (Make use of an ellipsis.)

$s_n =$

(b) Multiply your equation by r.

$r \cdot s_n =$

(c) Subtract your two equations.

(d) Solve your new equation for s_n.

(e) What assumption are you making about r?

(f) Compute $\lim_{n\to\infty} s_n$. For what values of r does this limit converge?

(g) For what values of r does $\sum_{k=0}^{\infty} r^k$ converge? To what?

Exercise 2.3.2. Compute the following.

(a) $\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k$

(b) $\sum_{k=0}^{\infty} \frac{5^k}{6^{k+1}}$

(c) $\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+2}}{3^{3k+3}}$

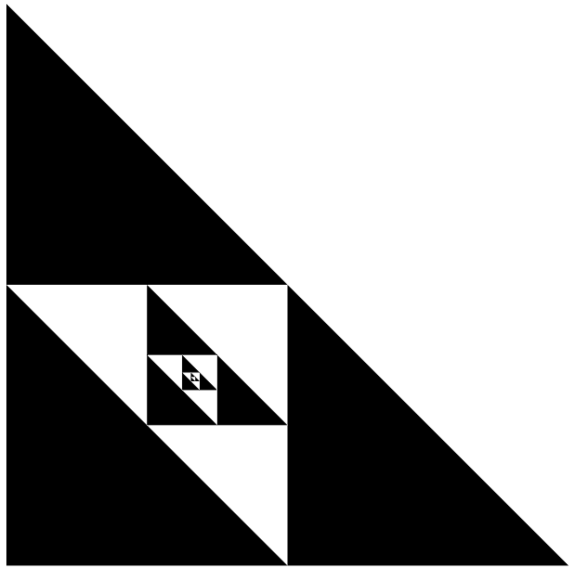
Exercise 2.3.3. Consider the figure below. Assume the largest triangle has an area of 1 unit. To get the second figure, the middle triangle is removed from the black triangle in the first figure. To get the third figure, the middle triangle is removed from each black triangle in the second figure. This process is repeated. The limiting figure is called the Sierpinski triangle.

[Image source: wikimedia]

(a) Determine the area of white region.

(b) Determine the area of the black region.

Exercise 2.3.4. Consider the figure below. Assume the largest triangle has base and height of 1 unit.

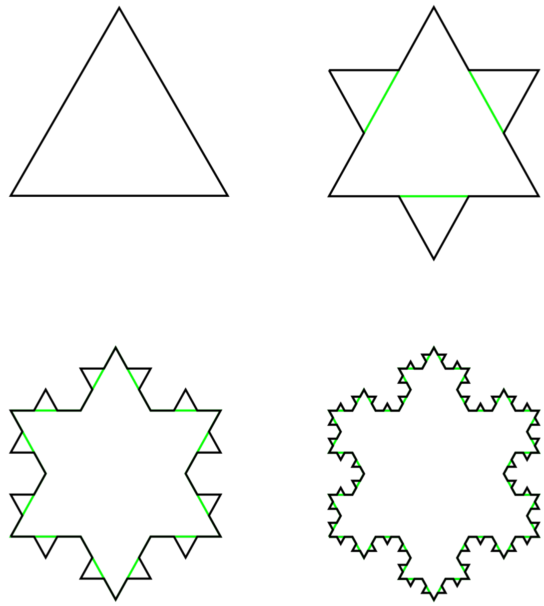


[Image source: I made this one.]

(a) Determine the area of black region.

(b) Determine the area of the white region.

Exercise 2.3.5. Consider the figure below. The limiting object is called the Koch snowflake.



[Image source: wikimedia]

(a) Determine the area enclosed by the Koch snowflake.

(b) Determine the perimeter of the Koch snowflake.

2.4 Tests for Series

Here is our first basic test for divergence of a series.

Theorem (Test for Divergence)**.** If $\sum_{k=0}^{\infty} a_k$ is a series for which $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum_{k=0}^{\infty} a_k$ diverges. Equivalently, if $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

Exercise 2.4.1. Check that each of the series below diverges by applying the Test for Divergence.

(a) $\sum_{k=0}^{\infty} k$

(b) $\sum_{k=1}^{\infty} \frac{k}{k+1}$

(c) $\sum_{j=3}^{\infty} \sin(j)$

(d) $\sum_{i=1}^{\infty} \ln(i)$

The cautionary tale surrounding the Test for Divergence is that the converse is not true. That is, there are series $\sum_{k=0}^{\infty} a_k$ which diverge even though $\lim_{k\to\infty} a_k = 0$. The standard example is the harmonic series.

Exercise 2.4.2. Let $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ be the **harmonic series**.

(a) Write out the partial sums s_1, s_2, s_4, s_8, and s_{16}. Do not simplify any of these sums.

(b) Which is bigger: $\frac{1}{3} + \frac{1}{4}$ or $\frac{1}{4} + \frac{1}{4}$? Don't actually add these fractions to answer this question.

(c) Which is bigger: $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$ or $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$? Don't actually add these fractions to answer this question.

(d) Which is bigger: $\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}$ or $\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$? Don't actually add these fractions to answer this question.

(e) Show that $s_{2^n} \geq 1 + \frac{n}{2}$.

(f) Compute $\lim_{n\to\infty} s_{2^n}$.

(g) Conclude that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges to infinity.

Exercise 2.4.3. Consider now the series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$.

(a) Have a discussion with your neighbor about whether or not the series converges.

(b) Write the series in the form $\sum_{k=0}^{\infty} a_k$.

(c) Compute $\lim_{k\to\infty} a_k$.

(d) What does the test for divergence tell you in this case?

(e) Does $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ converge?

Exercise 2.4.4. Construct two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ so that each diverges, and also $\sum_{k=1}^{\infty} (a_k + b_k)$ diverges.

Exercise 2.4.5. Construct two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ so that each diverges, but $\sum_{k=1}^{\infty} (a_k + b_k)$ converges.

Exercise 2.4.6. What happens in the case where both $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge? What if one converges and the other diverges?

Exercise 2.4.7. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p > 1$.

(a) Argue that the sequence of partial sums s_k is a monotonic increasing sequence. That is, that the partial sums get bigger as k gets bigger.

(b) Consider the partial sum $s_{2k+1} = \sum_{n=1}^{2k+1} \frac{1}{n^p}$.

(c) Show that $s_{2k+1} = 1 + \sum_{i=1}^{k} \left(\frac{1}{(2i)^p} + \frac{1}{(2i+1)^p} \right)$. (It may help to write out both sides for $k = 1, 2, 3$ to see what is happening here.

(d) Show that $s_{2k+1} < 1 + \sum_{i=1}^{k} \frac{2}{(2i)^p}$.

(e) Build on the previous step to show that $s_{2k+1} < 1 + 2^{1-p} s_k$.

(f) Build on the previous step to show that $s_{2k+1} < 1 + 2^{1-p} s_{2k+1}$

(g) Solve the previous inequality for s_{2k+1}.

(h) Conclude that s_k is a bounded sequence.

(i) Does s_k converge?

2.5 Tests for Series

Exercise 2.5.1. For each series below, apply each of the following tests and state the results of the test.

- The Test for Divergence
- The p-series Test
- The Geometric Series Test
- The Limit Comparison Test
- The Ratio Test

If a particular test does not apply to your series, say so. For the Limit Comparison Test, try to choose a "good" series to compare with.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-3}{n^2+3n-7}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3-5n^2+1}{6n^7-4n^4+23}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+7}}{5^{n-1}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 4^{2n}}{(n+1)^4 5^n}$

Recall that if n is a positive integer, then $n! = n(n-1)\cdots 3\cdot 2\cdot 1$, (e.g. $4! = 4\cdot 3\cdot 2\cdot 1 = 24$) and $0! = 1$.

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{10^n}{n!}$

Exercise 2.5.2. Compute the following limits. It will help to remember your tricks for computing limits from differential calculus.

(a) $\lim_{n\to\infty} \sqrt[n]{n}$.

(a) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

Exercise 2.5.3. Determine the convergence or divergence of the series below.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{n!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n}$

2.6 Tests for Series

A series is **alternating** if it can be written in the form $\sum_{n=0}^{\infty}(-1)^n a_n$ where $a_n \geq 0$. If we expand the sum, it looks like

$$\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \cdots$$

and so the terms being added alternate between positive and negative.

Exercise 2.6.1. Determine which of the series below are alternating.

(a) $\sum_{n=1}^{\infty} \frac{1}{n}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(c) $\sum_{n=1}^{\infty} \left(\frac{-1}{n}\right)^2$

(d) $\sum_{n=1}^{\infty} \frac{(-1)^{3n+2}}{n}$

(e) $\sum_{n=1}^{\infty} \frac{(-1)^{4n+5}}{n}$

(f) $\sum_{n=0}^{\infty}(-1)^n a_n$ where $a_n = (-1)^n$.

(g) $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots$

The nice thing about alternating series is that there is a simple test for convergence.

Theorem (Alternating Series Test)**.** If $\sum_{n=0}^{\infty}(-1)^n a_n$ is an alternating series for which $|a_n| \geq |a_{n+1}|$ for all n and $\lim_{n\to\infty} |a_n| = 0$, then the series $\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

Exercise 2.6.2. For each series in the previous exercise, decide if the Alternating Series Test applies. If it does, apply it and state the results.

Exercise 2.6.3. Determine for which values of p the series below converges and for which it diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$

Given an arbitrary series $\sum_{n=0}^{\infty} a_n$, if the series $\sum_{n=0}^{\infty} |a_n|$ converges, then we say the original series is **absolutely convergent**. Absolutely convergent series are convergent. If $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |a_n|$ diverges, then $\sum_{n=0}^{\infty} a_n$ is **conditionally convergent**.

Exercise 2.6.4. Determine for which values of p the series below is absolutely convergent and those for which it is conditionally convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$

Exercise 2.6.5. Determine the values of x for which the series below are absolutely convergent. A good tool to use is the Ratio Test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{2^{n+1}}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

2.7 Taylor Polynomials

Recall that a polynomial centered at a is one of the form

$$p(x) = \sum_{k=0}^{d} c_k (x-a)^k$$

for coefficients c_k.

Exercise 2.7.1. Consider the polynomial $p(x) = 1 - x + x^2$.

(a) What is the center of this polynomial?

(b) Find coefficients b_0, b_1, b_2 so that $p(x) = b_0 + b_1(x-1) + b_2(x-1)^2$. That is, recenter the polynomial at 1.

(c) Recenter $p(x)$ at -3.

We want an efficient way to recenter a polynomial.

Exercise 2.7.2. Suppose $p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots + c_d(x-a)^d$.

(a) Compute $p(a)$.

(b) Compute $p'(x)$.

(c) Compute $p'(a)$.

(d) Compute $p''(x)$.

(e) Compute $p''(a)$.

(f) Compute $p^{(3)}(x)$.

(g) Compute $p^{(3)}(a)$.

(h) Compute $p^{(4)}(x)$.

(i) Compute $p^{(4)}(a)$.

(j) Compute $p^{(5)}(x)$.

(k) Compute $p^{(5)}(a)$.

(l) Determine c_0 in terms of $p(a)$.

(m) Determine c_1 in terms of $p'(a)$.

(n) Determine c_2 in terms of $p''(a)$.

(o) Determine c_3 in terms of $p^{(3)}(a)$.

(p) Determine c_4 in terms of $p^{(4)}(a)$.

(q) Determine c_5 in terms of $p^{(5)}(a)$.

(r) Determine c_n in terms of $p^{(n)}(a)$ where n is any positive integer.

Exercise 2.7.3. Check that at the end of the previous exercise you got

$$c_n = \frac{p^{(n)}(a)}{n!}$$

Hence, if $p(x) = \sum_{k=0}^{d} c_k(x-a)^k$, then you can determine the coefficients by taking derivatives.

Exercise 2.7.4. Let $p(x) = x^5$.

(a) Recenter p at 1 by using the shortcut method involving derivatives.

(b) Recenter p at -2.

Given a function f, we want to approximate f at a with a polynomial function centered at a. Using our previous work as inspiration, we make the following definition:

Definition 2.7.5. Let f be n-times differentiable at a. i.e., $f^{(k)}(a)$ exists for $k = 0, 1, \ldots, n$. Then the **nth-order Taylor polynomial of f centered at a** is the polynomial

$$T_f^n(x;a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The hope is that for values of x near a, $T_f^n(x;a) \approx f(x)$.

Exercise 2.7.6. Let $f(x) = \exp(x)$.

(a) Compute $T_f^0(x;0)$, $T_f^1(x;0)$, $T_f^2(x;0)$, $\ldots$, $T_f^5(x;0)$. Generalize to $T_f^n(x;0)$

(b) Compute $T_f^k(0.2;0)$ for $k = 0, 1, \ldots, 5$ and compare with a numerical approximation for $\exp(0.2)$ provided by a calculator or some other computer. Do the approximations seem to be getting better? Explain.

Exercise 2.7.7. Repeat the previous problem with $g(x) = \sin(x)$.

2.8 Taylor Series

Recall that the Taylor series for f centered at a is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and that we are hoping for two things: first that the series converges (at least for values of x near a), and second that when the series does converge, it converges to $f(x)$. We have a couple of tools to help us with this. The first is Taylor's theorem.

Theorem (Taylor-Lagrange)**.** Suppose f has derivatives of all orders on an open interval I around a. Then for all x in I there is a c in I between x and a so that the Taylor remainder $R_f^n(x;a) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$.

In particular, the Taylor error is $E_f^n(x;a) = \left| \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right|$

Exercise 2.8.1. Let $f(x) = \cos(x)$.

(a) Compute the 5th order Taylor polynomial approximation for $f(x) = \cos(x)$ centered at 0.

(b) Apply the Taylor-Lagrange Theorem to determine $E_f^n(x;0)$. You should have an expression involving an unknown c.

(c) Determine a reasonable upper bound for this error. (Hint: What is the largest possible value that $|\cos(c)|$ or $|\sin(c)|$ could take?)

Two other useful observations are the following:

- If $|x| < 1$, then $\lim\limits_{n\to\infty} |x| = 0$.
- If x is any fixed real number, then $\lim\limits_{n\to\infty} \dfrac{|x|^n}{n!} = 0$.

Exercise 2.8.2. Compute $\lim\limits_{n\to\infty} E_f^n(x;0)$ where $f(x) = \cos(x)$.

The following theorem tells us that the Taylor error is exactly what we need to consider if we want to know if a function is represented by its Taylor series.

Theorem. A function f is represented by its Taylor series centered at a on an interval I if and only if $\lim\limits_{n\to\infty} E_f^n(x;a) = 0$ for all x in that interval. In this case we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Exercise 2.8.3. For what values of x is $f(x) = \cos(x)$ equal to its Taylor series centered at 0? Write out this series.

Exercise 2.8.4. Repeat the above for $g(x) = \sin(x)$.

Exercise 2.8.5. Consider the function $h(x) = \exp(x)$.

(a) Compute the Taylor series for h centered at 0.

(b) Compute $E_h^n(x; 0)$. Again, you should have an expression involving an unknown c.

(c) Recall that this c is between x and 0.

(d) If $x < 0$, which is bigger: 1, $\exp(x)$, or $\exp(c)$?

(e) If $x > 0$, which is bigger: 1, $\exp(x)$, or $\exp(c)$?

(f) Compute $\lim_{n\to\infty} E_h^n(x; 0)$.

(g) For what values of x is $\exp(x)$ equal to its Taylor series?

Exercise 2.8.6. Repeat the previous process for $f(x) = \dfrac{1}{1-x}$ centered at 0.

2.9 Differentiation and Anti-differentiation of Power Series

One cool thing about power series is that if we have a function f and represent it by a power series, then we can also represent f', the derivative of f, as a power series. Even better, we can use what we know about the power series for f to obtain the power series for f'. So, if we have

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

we can differentiate "term by term" to get

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$$

and the radius of convergence for the series for f and f' are the same.

Exercise 2.9.1. Let $f(x) = \dfrac{1}{1-x}$.

(a) Compute $f'(x)$.

(b) Write down the Taylor series for f centered at 0. (This is one you should know. So if you don't, look it up.)

(c) Determine a power series representation for $f'(x)$.

One thing to notice is that if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

then

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$$

but the exponent on $(x-a)$ in this expression is $n-1$. We would rather it be n instead. With the right re-indexing, this can be fixed.

Exercise 2.9.2. Consider the series $f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$.

(a) Let $m = n - 1$. Solve for n.

$n =$

(b) Replace all n's in the series above with what you just found.

(c) Now take your series with m's and replace all the m's with n's. (We can do this, because the m and n are just dummy variables. They could just as well be k's or ℓ's or i's or j's.)

(d) Notice your new series starts at $n = -1$. Explain why we can start at $n = 0$ instead and not change the value of the series.

Exercise 2.9.3. Let $f(x) = \dfrac{1}{1+x^2}$.

(a) Determine the Taylor series for f centered at 0 by making a substitution for the one for $\dfrac{1}{1-x}$.

(b) Determine $f'(x)$ and a power series for f'.

We can do the same trick with series for anti-derivatives. That is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

then

$$\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

where C is our old friend the constant of integration. Notice that now we have $n+1$ as an exponent. We don't like that.

Exercise 2.9.4. (a) Reindex the series $\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$ so that the exponent on $(x-a)$ is n.

(b) Notice that your new series begins at $n = 1$. Why is that?

Exercise 2.9.5. Let $f(x) = \arctan(x)$.

(a) Compute $f'(x)$.

(b) Determine a power series for $f'(x)$. (Hint: You already did this.)

(c) Use your series for f' to get a series for f.

(d) Determine the appropriate value for C.

Exercise 2.9.6. Determine a power series for $f(x) = \ln(1 + x)$.

Exercise 2.9.7. Determine a power series for $f(x) = x\cos(x) + \sin(x)$.

2.10 Riemann Sums

Recall that the definite integral of f on $[a, b]$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 1, \ldots, n$. This definite integral represents the area under the curve f from a to b.

Exercise 2.10.1. Let $f(x) = x + 1$ and $[a, b] = [-1, 2]$.

(a) Make a sketch which indicates what $\displaystyle\int_{-1}^{2} f(x)\,dx$ represents.

(b) Determine Δx.

(c) Determine x_i.

(d) Determine the Riemann sum $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$. Simplify it.

(e) If you didn't already, make use of the fact that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and that $\sum_{i=1}^{n} 1 = n$ to further simplify your Riemann sum.

(f) Compute $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$.

(g) Compute $\int_{-1}^{2} x + 1\, dx$.

(h) Compare your result with your sketch. Does it make sense?

Exercise 2.10.2. Repeat the previous exercise for the function $g(x) = x^2$ on the interval $[0, 2]$. That is, compute $\int_0^2 x^2\, dx$. You may need to make use of the fact that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Exercise 2.10.3. Compute $\int_{-2}^{2} x^2\, dx$. (*Hint*: Draw a picture and use your result above.

Exercise 2.10.4. (a) Compute $\int_{-1}^{1} x^2 - 1\, dx$.

(b) Is your result positive or negative?

(c) Why does this make sense?

Exercise 2.10.5. Suppose $\int_0^3 f(x)\,dx = 0$. Describe what this means in terms of area under f from 0 to 3. Draw some pictures to illustrate the possibilities.

Exercise 2.10.6. Determine all values for b so that $\int_0^b 1 - 3x^2\,dx = 0$. Sketch a picture to explain each case.

2.11 One Tricky Definite Integral

Here you will compute $\int_0^{\pi} \sin(x)\, dx$ using the definition of the Riemann integral.

Exercise 2.11.1. Consider $\int_0^{\pi} \sin(x)\, dx$.

(a) Check that in this case $x_i = i\Delta x$. (For what we are doing, we want everything in terms of Δx.)

(b) Write the Riemann sum $\sum_{i=1}^{n} \sin(x_i)\Delta x$.

The rest of the work will be to massage this Riemann sum to the point where we can take a limit to evaluate the definite integral.

(c) Write down the sum of angles formula for the cosine function. (Look this up if you have to.)

$\cos(A + B) =$

(d) Write down the difference of angles formula for the cosine function. (Look this up if you have to.)

$\cos(A - B) =$

(e) Use the above to simplify $\cos(A - B) - \cos(A + B)$.

(f) Replace $A - B$ with a and $A + B$ with b and simplify.

(g) Check that you have $\cos(b) - \cos(a) = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$.

(h) Plug in $a = \frac{2i+1}{2}\Delta x$ and $b = \frac{2i-1}{2}\Delta x$ and simplify.

(i) You should have an equation involving $\sin(i\Delta x)$. Solve your equation for this term.

(j) Use your last result to rewrite the Riemann sum under investigation: $\sum_{i=1}^{n} \sin(x_i)\Delta x$.

(k) Factor out all terms which do not contain i. You should be left with a difference of two cosine terms in the summation.

(l) Evaluate that sum. (*Hint*: It is a telescoping sum. Write out some of the terms to see this. The whole point of all of the previous work was to get our sum to look like a telescoping sum.)

(m) Recall that $\int_0^{\pi} \sin(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sin(x_i)\Delta x$.

(n) If $n \to \infty$, what happens to Δx?

(o) Express $\int_0^{\pi} \sin(x)\, dx$ as a limit involving Δx. That is, replace $\lim_{n\to\infty}$ with $\lim_{\Delta x \to c}$ for the appropriate value of c.

(p) Evaluate the limit.

(q) What is $\int_0^{\pi} \sin(x)\, dx$?

(r) Draw a picture illustrating what this definite integral represents.

2.12 The Fundamental Theorem of Calculus, Part 1

Theorem (FTC1)**.** Suppose f is continuous on an interval containing a. Then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

on that interval.

Exercise 2.12.1. Compute the following.

(a) $\displaystyle \frac{d}{dx}\int_0^x t^2 - t + 3\,dt$

(b) $\displaystyle \frac{d}{dy}\int_\pi^y \sin(x)\,dx$

(c) $\displaystyle \frac{d}{dx}\int_{-17}^x \frac{\exp(s)}{42s^4 + s^2 + 1}\,ds$

(d) $\displaystyle \frac{d}{dr}\int_0^r |\theta|\,d\theta$

(e) $\displaystyle \frac{d}{dx}\int_0^x \gamma\,d\gamma$

(f) $\displaystyle \frac{d}{dx}\int_0^x \alpha\,d\alpha$

Exercise 2.12.2. Compute the following:

$$\frac{d}{dx}\int_0^{x^2} t^2 - t + 3\ dt \text{ (\textit{Hint}: Chain Rule)}$$

$$\frac{d}{dy}\int_y^{\pi} \sin(x)\ dx$$

$$\frac{d}{dx}\int_{-17x}^{x} \frac{\exp(s)}{42s^4 + s^2 + 1}\ ds \text{ (\textit{Hint}: Separate into two integrals first.)}$$

$$\frac{d}{dr}\int_{\sin(r)}^{r} |\theta|\ d\theta$$

Some Special Functions

Exercise 2.12.3. Let f be the function defined below.

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(a) Show f is continuous at 0.

(b) Is f integrable on $\mathbb{R}$? Explain.

(c) Define Si to be the function below.

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\, dt$$

This is the **sine integral** function. Compute $\mathrm{Si}'(x)$.

(d) Let $g(x) = x^2\,\mathrm{Si}(x)$. Compute $g'(x)$.

(e) Let $h(x) = \mathrm{Si}(x^2)$. Compute $h'(x)$.

(f) Determine (or just recall) the Taylor series for $\sin(x)$ centered at 0 and its radius and interval of convergence.

(g) Determine the Taylor series for $\dfrac{\sin(x)}{x}$ centered at 0 and its radius and interval of convergence.

(h) Determine the Taylor series for $\text{Si}(x)$ centered at 0 and its radius and interval of convergence.

Exercise 2.12.4. Let erf, the so-called **error function**, be defined as below:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt$$

(a) Compute $\operatorname{erf}'(x)$.

(b) Let $g(x) = x^2 \operatorname{erf}(x)$. Compute $g'(x)$.

(c) Let $h(x) = \operatorname{erf}(x^2)$. Compute $h'(x)$.

(d) Determine (or just recall) the Taylor series for $\exp(x)$ centered at 0 and its radius and interval of convergence.

(e) Determine the Taylor series for $\frac{2}{\sqrt{\pi}} \exp(-x^2)$ centered at 0 and its radius and interval of convergence.

(f) Determine the Taylor series for $\text{erf}(x)$ centered at 0 and its radius and interval of convergence.

2.13 Logarithms

We computed (a long time ago) that $\int \frac{1}{x}\, dx = \ln|x| + C$. When we did this, we made use of the fact that we defined the natural log function ln to be the inverse function to the exponential function exp. One alternative way to define the natural logarithm is by using integrals.

Let's explore this setup now. Define ln to be the function defined on $(0, \infty)$ by

$$\ln(x) = \int_1^x \frac{1}{t}\, dt$$

Exercise 2.13.1. Use the definition of ln above to compute $\ln(1)$.

Exercise 2.13.2. Is ln differentiable? Explain.

Exercise 2.13.3. Is ln continuous? Explain.

Exercise 2.13.4. Use a graph to explain why ln is an increasing function.

Exercise 2.13.5. Compute $\lim_{x\to\infty} \ln(x)$ as follows.

(a) Sketch a graph of $y = \dfrac{1}{x}$.

(b) Which is bigger: $\dfrac{1}{2}$ or $\ln(2)$? Determine this by looking at your graph and considering areas.

(c) Which is bigger: $\dfrac{1}{2} + \dfrac{1}{3}$ or $\ln(3)$? Determine this by looking at your graph and considering areas.

(d) Which is bigger: $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5}$ or $\ln(5)$?

(e) Which is bigger: $\displaystyle\sum_{i=2}^{n} \frac{1}{i}$ or $\ln(n)$?

(f) Compute $\lim_{n\to\infty} \ln(n)$ by comparing with what you know about $\displaystyle\lim_{n\to\infty} \sum_{i=2}^{n} \frac{1}{i}$.

Exercise 2.13.6. Compute $\lim_{x \to 0+} \ln(x)$ using your work above and some geometry.

Exercise 2.13.7. From what you have done in the past few exercises, what is the range of ln?

Exercise 2.13.8. Let x and a be real numbers. You will show that $\ln(ax) = \ln(a) + \ln(x)$.

(a) Compute $\frac{d}{dx} \ln(ax)$ from the definition of ln. (Recall that a is a constant.)

(b) Compute $\frac{d}{dx} (\ln(a) + \ln(x))$ from the definition of ln. (Recall that a is a constant.)

(c) Check that you have the same thing in each case.

(d) What does this tell you about the relationship between $\ln(ax)$ and $\ln(a) + \ln(x)$?

(e) Determine the constant.

Exercise 2.13.9. Use a similar technique to show that $\ln(x^a) = a\ln(x)$.

Exercise 2.13.10. Consider the equation $\ln(x) = 1$.

(a) Based on the work in the above exercises, does the equation have a solution? Explain.

(b) How many solutions does the equation have? Explain.

(c) This unique solution gets a special name: e. That is, e is the unique real number such that $\ln(e) = 1$. Write this defining equation for e as an integral.

Exponentials

Exercise 2.13.11. Is ln invertible? Explain.

Exercise 2.13.12. Let exp be the inverse function to ln.

(a) Determine the domain and range of exp.

(b) Compute $\frac{d}{dx}\exp(x)$ as follows:

- (i) Let $y = \exp(x)$. Solve for x.

- (ii) Take the derivative of the equation you just found using implicit differentiation.

- (iii) Solve for y' and replace y with $\exp(x)$.

Some Definite Integrals

Exercise 2.13.13. Compute these.

(a) $\int_0^1 \exp(x)\, dx.$

(b) $\int_1^e \ln(x)\, dx.$

(c) $\int_0^{\pi/4} \tan(x)\, dx$

(d) $\int_{-1}^{1} x \exp(x^2)\, dx$.

(e) $\int_{-1}^{1} x^2 \exp(x)\, dx$.

(f) $\int_{1}^{e} x \ln(x)\, dx$

2.14 Improper Integrals

Recall that an integral $\int_a^b f(x)\,dx$ is **improper** if either f is unbounded on the interval (a,b) or either a or b is $\pm\infty$. Improper integrals need to be resolved by computing limits. If all of the appropriate limits converge, then the improper integral converges. Otherwise, the improper integral diverges.

Exercise 2.14.1. Consider $\int_0^2 \frac{1}{x^2}\,dx$.

(a) This integral is improper. Why?

(b) What limit needs to be computed to decide if this integral converges or diverges?

(c) Compute that limit.

(d) What is $\int_0^2 \frac{1}{x^2}\,dx$?

Exercise 2.14.2. Consider $\int_{-1}^{1} \frac{1}{x^2}\,dx$.

(a) This integral is improper. Why?

(b) What limits need to be computed to decide if this integral converges or diverges? (There are two of them.)

(c) Compute those limits.

(d) What is $\int_{-1}^{1} \frac{1}{x^2}\,dx$?

Exercise 2.14.3. Determine whether or not $\int_0^2 \frac{1}{x^2-1}\, dx$ converges or diverges. If it converges, determine what it converges to. (A partial fraction decomposition may be helpful.)

Exercise 2.14.4. Determine whether or not $\int_2^\infty \frac{1}{x^2-1}\, dx$ converges or diverges. If it converges, determine what it converges to. (A partial fraction decomposition may be helpful.)

Exercise 2.14.5. (a) Determine whether or not $\int_{-\infty}^{0} \exp(x)\ dx$ converges or diverges. If it converges, determine what it converges to.

(b) Determine whether or not $\int_{0}^{1} \ln(x)\ dx$ converges or diverges. If it converges, determine what it converges to.

(c) Why are the previous two integrals opposite? (Hint: Look at a graph.)

Exercise 2.14.6. The function $\Gamma(n) = \int_0^\infty \exp(-x)x^{n-1}\,dx$ is called the **gamma function**.

(a) Compute $\Gamma(1)$, $\Gamma(2)$, and $\Gamma(3)$.

(b) Show that $\Gamma(n+1) = n\Gamma(n)$ for all n.

(c) Use the last part to compute $\Gamma(n)$ for $n = 1, 2, \ldots, 10$.

Some Review

Exercise 2.14.7. Choose an integer n between 1 and 16. Make sure your choice is different from the one your neighbors pick. Write your choice below.

$$n =$$

Exercise 2.14.8. Choose an odd integer m between 1 and 9. Write your choice below.

$$m =$$

Exercise 2.14.9. Write out the numbers $n + km$ for integer values of $k = 0, \ldots, 15$. That is, write out n, $n + m$, $n + 2m$, $n + 3m$, etc.

Exercise 2.14.10. For each of your numbers above, if it is greater than 16, subtract 16 from it until it is less than or equal to 16. For example, if you have 47, then subtract 16 to get 31, and subtract 16 again to get 15.

Write your new list of numbers below.

Exercise 2.14.11. For each number on your list, create an example exercise which measures the learning outcome below with that number. Proceed in order in your list.

1. State the definition of infinite series, infinite sums, and partial sums, and compute partial sums of a series.
2. State the definition of convergent and divergent series.
3. Compute the sum of convergent telescoping series.
4. State and apply the divergence test for series.
5. State and apply algebraic properties of convergent series.
6. State and apply the p-series test, the comparison test, the limit comparison test, the integral test, the ratio test, and the root test.
7. State the definitions of conditionally and absolutely convergent series and apply appropriate tests to determine if a series is conditionally or absolutely convergent.
8. Determine Taylor polynomials and Taylor series for elementary functions and their derivatives and anti-derivatives.
9. Estimate the error in approximating a function by it's Taylor polynomial.
10. Determine the radius and interval of convergence for a power series.
11. State the definition of a Riemann sum and use it to approximate area under a curve.
12. State the definition of a definite integral and compute values of definite integrals using sequence and series techniques.
13. State and apply algebraic properties of definite integrals.

14. State and apply the mean value theorem for integrals.
15. State the Fundamental Theorem of Calculus and use it to compute definite integrals.
16. Compute improper integrals.

2.15 Volumes

Exercise 2.15.1. Let R be the region in the first quadrant bounded by the curve $y = x^2$, the y-axis, and the line $y = 1$.

(a) Sketch the region R.

(b) Sketch the solid obtained by rotating R around the y-axis.

(c) Slice the solid into discs and determine the volume dV of a typical disc. Draw this disc.

(d) Determine the volume V of the object.

Exercise 2.15.2. Repeat the previous exercise for the same region, but revolved around the specified axis.

(a) The x-axis.

(b) The line $x = 1$.

(c) The line $y = 1$.

(d) The line $y = 2$.

Exercise 2.15.3. Let R be the region in the first quadrant between the curves $y = x$ and $y = x^2$. Determine the volume of the solid obtained by revolving R about the specified axis.

(a) The line $y = 1$.

(b) The line $x = 1$.

(c) The line $y = x$.

(d) The line $y = x + 1$.

2.16 Polar Coordinates

Recall that a point in $\mathbb{R}^2$ with Cartesian coordinates (x, y) can be described with polar coordinates (r, θ) where

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

$$x^2 + y^2 = r^2$$

$$\theta = \begin{cases} \arctan(y/x) & \text{if } x > 0 \\ \arctan(y/x) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \pi/2 & \text{if } x = 0 \text{ and } y > 0 \\ -\pi/2 & \text{if } x = 0 \text{ and } y < 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

Exercise 2.16.1. Convert the Cartesian coordinates below to polar coordinates.

(a) $(0, 4)$

(b) $(1, 1)$

(c) $(-2, \sqrt{2})$

(d) $(-4, -4)$

(e) $(0, -3)$

(f) $(2, -2)$

Exercise 2.16.2. Convert the polar coordinates below to Cartesian coordinates.

(a) $(1, 0)$

(b) $(3, \pi/4)$

(c) $(6, \pi/2)$

(d) $(2, 5\pi/6)$

(e) $(0, \pi)$

(f) $(3, 7\pi/4)$

Exercise 2.16.3. Sketch the regions in the plane described by the inequalities below.

(a) $r \leq 3$

(b) $r \geq 2$ and $\pi \leq \theta \leq 3\pi/2$

(c) $1 \leq r \leq 2$

(d) $0 \leq \theta \leq \pi/4$ and $0 \leq r \leq \sec(\theta)$.

Exercise 2.16.4. Sketch a graph of the polar equation $r = \cos(n\theta)$ for $n = 0, 1, 2, 3, 4, 5, 6$. Conjecture about the shape of the graph for a general positive integer n.

Exercise 2.16.5. For finding areas enclosed by polar curves, we will need to recall the area of a sector of a circle. Recall a sector of a circle is the region inside a circle bounded between two radii separate by an angle θ.

(a) Sketch a sector of a circle or radius r with angle θ.

(b) Determine the area of a sector of radius r and angle π.

(c) Determine the area of a sector of radius r and angle $\pi/2$.

(d) Determine the area of a sector of radius r and angle $\pi/3$.

(e) Determine the area of a sector of radius r and angle $\pi/4$.

(f) Determine the area of a sector of radius r and angle θ.

Geometry

Exercise 2.16.6. For each problem below, do the following:

(i) Sketch the curve on the given domain.

(ii) Set up an integral which represents the length of the curve on that domain.

(iii) Sketch the region R under the curve on that domain.

(iv) Set up an integral which represents the area of the region R.

(v) Sketch the solid region M obtained by rotating R about the specified axis.

(vi) Set up an integral which represents the volume of the solid region M.

(vii) Set up an integral which represents the surface area of M.

(a) $y = x^2$, $[1, 3]$, the x-axis.

(b) $y = \exp(x)$, $[-1, 1]$, the line $x = 1$.

(c) $y = \sqrt{x+1}$, $[-1, 0]$, the y-axis.

(d) $y = \ln(x)$, $[1, e]$, the line $y = 1$.

(e) $y = \cos(x)$, $[-\pi/2, \pi/2]$, the x-axis.

(f) $y = 1 - x^2$, $[-1, 1]$, the line $y = 2$.

Review: Sequences, Sums, Series

Exercise 2.16.7. Determine the values of each of the expressions below.

(a) $\sum_{n=1}^{700} 3$

(b) $\sum_{n=1}^{100} n$

(c) $\sum_{n=1}^{345} n^2$

(d) $\sum_{n=1}^{123} n^3$

(e) $\sum_{n=1}^{42} 4n^2 - 3n + 5$

Exercise 2.16.8. Determine the limits below. If they diverge, explain why.

(a) $\lim_{n\to\infty} \frac{4n^2 - 3n + 7}{3 - \frac{4}{3}n^2 + 3n}$

(b) $\lim_{n\to\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$

(c) $\lim_{n\to\infty} \frac{\ln(n)}{n}$

(d) $\lim_{n\to\infty} \frac{\exp(n)}{n^{700}}$

(e) $\lim_{x \to 0^+} \ln(x)$

(f) $\lim_{x \to \infty} \ln(x)$

(g) $\lim_{x \to \infty} \exp(x)$

(h) $\lim_{x \to -\infty} \exp(x)$

(i) $\lim_{x \to \infty} \arctan(x)$

(j) $\lim_{x \to -\infty} \arctan(x)$

Exercise 2.16.9. Determine whether or not the series below converge. Explain.

(a) $\sum_{n=0}^{\infty} \frac{4}{3^n}$

(b) $\sum_{n=0}^{\infty} \frac{4^n}{3}$

(c) $\sum_{n=0}^{\infty} \frac{4^n}{3^n}$

(d) $\sum_{n=0}^{\infty} (-1)^n \frac{4}{3^n}$

(e) $\sum_{n=0}^{\infty} (-1)^n \frac{4^n}{3}$

(f) $\sum_{n=0}^{\infty} (-1)^n \frac{4^n}{3^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+7}}{3n^3}$

(k) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$

(l) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$

(m) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n+2}}$

(n) $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n+7}}{3n^3}$

(o) $\sum_{n=0}^{\infty} \frac{3^n}{n!}$

(p) $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{(2n)!}$

(q) $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)}$

(r) $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

(s) $\sum_{n=0}^{\infty} \frac{\sin(n)}{n^2}$

Review: Taylor Series

Exercise 2.16.10. Determine the Taylor series for each of the functions with the specified center.

(a) $\exp(x)$, 0

(b) $\sin(x)$, 0

(c) $\cos(x)$, 0

(d) $\exp(x^2)$, 0

(e) $\int \exp(x^2)\, dx$, 0

(f) $\dfrac{1}{1-x}$, 0

(g) $\dfrac{1}{1+x}$, 0

(h) $\dfrac{1}{1+x^2}$, 0

(i) $\arctan(x)$, 0

(j) $\int \arctan(x)\, dx$, 0

(k) $\arctan(4x^2)$, 0

(l) $\int \arctan(4x^2)$, 0

(m) $\exp(x)$, 1

(n) $\sin(x)$, $\pi/2$

(o) $\sin(x)$, $\pi/4$

(p) $\arcsin(x)$, 0

(q) $\arccos(x)$, 0