MATH141A
Calculus & Analytic Geometry I
Workbook

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Contents

Introduction

This document is a compilation of worksheets I have created for use in a first semester calculus course. My course met five days a week for 50 minutes per class. The weekly routine was (roughly) worksheet, lecture, computer lab, worksheet, lecture. That is, the worksheets were separated by class periods during which I lectured. Below, if a worksheet seems to be assuming knowledge of content which has not been covered by previous worksheets, it is likely that this content was covered in lecture. Also, each worksheet reference sections of a textbook. For my class, we used "Worldwide Differential Calculus" by David B. Massey and published by Worldwide Center of Mathematics, and all section references below are to this textbook. Of course, these worksheets can be useful in this study of calculus in the absence of this, or any, textbook.

If you notice any typos or other errors, or are interested in my LaTeX code, please send me an email: wkronholm@whittier.edu. I would also appreciate hearing from instructors who use these materials in their courses.

Chapter 1

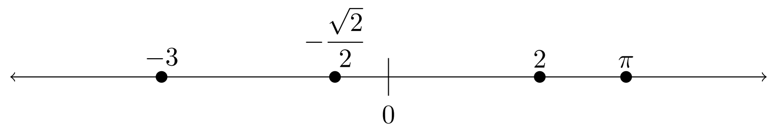
Limits & Continuity

1.1 Some Topology in $\mathbb{R}$

This worksheet discusses material corresponding roughly to section 1.3.1 of your textbook.

$\mathbb{R}$

The set $\mathbb{R}$ of real numbers should already be familiar to you. Sometimes we use interval notation and write $\mathbb{R} = (-\infty, \infty)$. We can picture $\mathbb{R}$ as a "number line" in the usual way:



The elements of $\mathbb{R}$ are the real numbers. That is, 4, π, $\sqrt{1010}$, 0, and -12.76 are all real numbers while $\sqrt{-1}$, $\frac{1}{0}$, and ∞ are not. We would write this with the following mathematical notation:

$$\pi \in \mathbb{R}$$

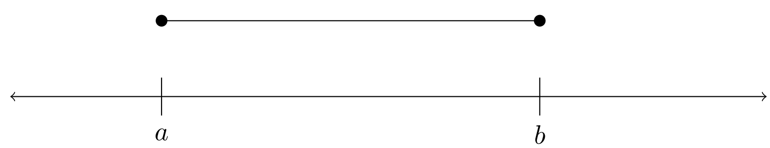
$$\sqrt{-1} \notin \mathbb{R}$$

You should read the first statement as "π is in $\mathbb{R}$" or "π is a member of the set of all real numbers" or "π is an element of the set $\mathbb{R}$." The second statement can be read as "$\sqrt{-1}$ is not a real number."

The following interval notation should also be familiar to you.

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

This is the **closed interval** from a to b. It is the set of all real numbers which are between a and b, or are equal to either a or b. We typically sketch this set on the number line as shown below.



Exercise 1.1.1. Decide which of the following statements are true. Draw a picture to illustrate your response.

(a) $4 \in [1, 7]$

(b) $-2 \in [-\sqrt{2}, 0]$

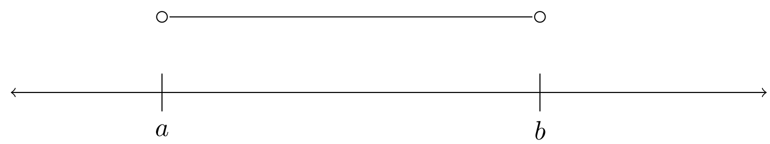
(c) $3 \in [e, \pi]$

(d) $0 \in [0, 101]$

Similarly, we have the **open interval** from a to b.

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

It is the set of all real numbers which are between a and b, but which are not equal to either a or b. We typically sketch this set on the number line as shown below.



Exercise 1.1.2. Decide which of the following statements are true. Draw a picture to illustrate your response.

(a) $4 \in (1, 7)$

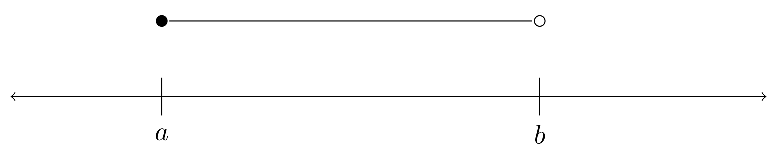
(b) $-2 \in (-\sqrt{2}, 0)$

(c) $3 \in (e, \pi)$

(d) $0 \in (0, 1)$

Also, we have the **half-open intervals**.

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$



$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

Exercise 1.1.3. (a) Find an open interval containing 10.

(b) Find a closed interval containing $\sqrt[3]{3}$.

(c) Find a half-open interval containing -43.69876234.

We also have the following infinite intervals.

$$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$$

$$(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$$

$$(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$$

$$[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$$

$$(-\infty, \infty) = \mathbb{R}$$

Exercise 1.1.4. Draw pictures of the above intervals.

The first two of the intervals above are open, the second two are closed, and the third is both open and closed. In general, a set X of real numbers is **open** if given any real number $a \in X$ there is an open interval containing a which lies entirely within X.

Exercise 1.1.5. Determine if the set indicated in the picture below is open. Explain.

Exercise 1.1.6. Determine if the set indicated in the picture below is open. Explain.

Given an interval of the form (a, b), $[a, b)$, $(a, b]$, or $[a, b]$, the **width** of the interval is $b - a$. If either $a = -\infty$ or $b = \infty$, then the interval is **infinitely wide**.

Exercise 1.1.7. Determine the width of each interval below.

(a) $[3, 10)$

(b) $(3, 10)$

(c) $(-\infty, 1)$

(d) $[-\sqrt{2}, \sqrt{2}]$

Exercise 1.1.8. (a) Find an open interval of width $1/2$ containing 3.1.

(b) Find a closed interval of width $1/10$ containing 0.

(c) Find an open interval of width $2/101$ containing -2.

(d) Find an open interval of width ϵ containing a.

Exercise 1.1.9. Consider the function $f(x) = 2x + 1$.

(a) Sketch a graph of f.

(b) Locate the point $(2, 5)$ on your graph.

(c) Suppose you want the output of the function f to be within 1 of 5. How close to 2 does the input need to be?

(d) Indicate on your graph your results from the previous exercise.

(e) Suppose you want the output of the function f to be within 1/2 of 5. How close to 2 does the input need to be?

(f) Indicate on your graph your results from the previous exercise.

(g) Suppose you want the output of the function f to be within 1/10 of 5. How close to 2 does the input need to be?

(h) Indicate on your graph your results from the previous exercise.

1.2 Intro to Limits

This worksheet discusses material corresponding roughly to section 1.3 of your textbook.

Limits are a way to examine the behavior of a function f near a point a. We begin with a typical example.

Exercise 1.2.1. Let $f(x) = 3x + 2$.

(a) Sketch a graph of f below.

(b) Mark the point $(1, f(1))$ on the graph.

(c) Complete the table below. (You may use a calculator to help if you wish.)

a	0.8	0.9	0.95	0.975	0.99	0.9984
$f(a)$						

(d) Do the values seem to be getting close to any particular number? If so, which one?

(e) Complete the table below. (You may use a calculator to help if you wish.)

a	1.3	1.1	1.06	1.04	1.0025	1.000117
$f(a)$						

(f) Do the values seem to be getting close to any particular number? If so, which one?

(g) Are the numbers in parts (d) and (f) the same? If so, write that number on the right of the expression below. Otherwise, write "DNE" instead.

$$\lim_{x \to 1} f(x) =$$

(h) Indicate the behavior you are observing on the graph you drew.

Exercise 1.2.2. Let g be the function below.

$$g(x) = \begin{cases} 3x + 2 & \text{if } x \neq 1 \\ -2 & \text{if } x = 1 \end{cases}$$

(a) Sketch a graph of g below.

(b) Mark the point $(1, g(1))$ on the graph.

(c) Complete the table below. (You may use a calculator to help if you wish.)

a	0.8	0.9	0.95	0.975	0.99	0.9984
$g(a)$						

(d) Do the values seem to be getting close to any particular number? If so, which one?

(e) Complete the table below. (You may use a calculator to help if you wish.)

a	1.3	1.1	1.06	1.04	1.0025	1.000117
$g(a)$						

(f) Do the values seem to be getting close to any particular number? If so, which one?

(g) Are the numbers in parts (d) and (f) the same? If so, write that number on the right of the expression below. Otherwise, write "DNE" instead.

$$\lim_{x \to 1} g(x) =$$

(h) Indicate the behavior you are observing on the graph you drew.

Exercise 1.2.3. Let h be the function below.

$$h(x) = \begin{cases} 3x+2 & \text{if } x \geq 1 \\ -2x & \text{if } x < 1 \end{cases}$$

(a) Sketch a graph of h below.

(b) Mark the point $(1, h(1))$ on the graph.

(c) Complete the table below. (You may use a calculator to help if you wish.)

a	0.8	0.9	0.95	0.975	0.99	0.9984
$h(a)$						

(d) Do the values seem to be getting close to any particular number? If so, which one?

(e) Complete the table below. (You may use a calculator to help if you wish.)

a	1.3	1.1	1.06	1.04	1.0025	1.000117
$h(a)$						

(f) Do the values seem to be getting close to any particular number? If so, which one?

(g) Are the numbers in parts (d) and (f) the same? If so, write that number on the right of the expression below. Otherwise, write "DNE" instead.

$$\lim_{x \to 1} h(x) =$$

(h) Indicate the behavior you are observing on the graph you drew.

Here is our formal definition of a limit:

Definition 1.2.4. Let f be a function which is defined for all input values near a, but not necessarily at a. Then we write

$$\lim_{x \to a} f(x) = L$$

and say "The limit of f as x approaches a is equal to L" when the following is true for a given number L:

For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

If no such L exists, then we say the limit does not exist and write $\lim_{x \to a} f(x)$ DNE.

The way to think about this definition is as follows: $\lim_{x \to a} f(x) = L$ means that we can make the value of $f(x)$ as close to L as we want by making x sufficiently close to a.

Let's return to the first example.

Exercise 1.2.5. Let $f(x) = 3x + 2$ and $a = 1$.

(a) Check that on the first page you had $\lim_{x \to 1} f(x) = 5$.

(b) Say we want to make the distance between $f(x)$ and 5 less than 2. Write an inequality which captures this idea.

(c) Solve this inequality for $|x - 1|$. That is, do algebra until you have an expression of the form $|x - 1| < \delta$ for some number δ.

(d) Complete this sentence: If x is within ______ of 1, then $f(x)$ is within ______ of 5.

(e) Repeat similar steps as above to complete this sentence: If x is within ______ of 1, then $f(x)$ is within $\dfrac{1}{10}$ of 5.

Exercise 1.2.6. Let $g(x) = x^2$. How close to 0 does x need to be to ensure that $g(x)$ is within 1 of $g(0)$? Within 1/10? Within 1/100? Within ϵ? What does this suggest about $\lim_{x\to 0} x^2$?

Exercise 1.2.7. Let $h(x) = x^3$. How close to 0 does x need to be to ensure that $h(x)$ is within 1 of $h(0)$? Within 1/10? Within 1/100? Within ϵ? What does this suggest about $\lim_{x\to 0} x^3$?

Exercise 1.2.8. Draw the graph of a function q which satisfies all of the conditions below.

- $\lim_{x\to 0} q(x) = 2$
- $\lim_{x\to 1} q(x) = -1$
- $\lim_{x\to -1} q(x)$ DNE
- $\lim_{x\to -2} q(x) = 1$
- $q(0) = 2$
- $q(1) = 1$
- $q(-1) = 0$
- $q(-2)$ DNE

1.3 Limits of Functions

This worksheet discusses material corresponding roughly to section 1.3 of your textbook.

Recall the following properties of limits.

- $\lim_{x\to a} c = c$ for any constant c.
- $\lim_{x\to a} x = a$.
- $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$, provided both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist.
- $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$ for any constant c, provided $\lim_{x\to a} f(x)$ exists.

We also have the following additional properties of limits.

- $\lim_{x\to a} (f(x)g(x)) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$, provided $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist.
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exists, and that $\lim_{x\to a} g(x) \neq 0$.

Exercise 1.3.1. Compute the limit $\lim_{x\to 2} 3x^2 - x + 4$ using the limit properties above.

Exercise 1.3.2. Compute the limit $\lim_{x\to 2} \frac{3x^2 - x + 4}{x + 1}$ using the limit properties above.

Exercise 1.3.3. Consider the functions f and g below.

$$f(x) = \begin{cases} 2x & \text{if } x > 1 \\ 2 & \text{if } x \le 1 \end{cases}$$

$$g(x) = \begin{cases} -2 & \text{if } x > 1 \\ -x - 1 & \text{if } x \le 1 \end{cases}$$

(a) Sketch graphs of f and g.

(b) Determine the limits $\lim_{x \to 1} f(x)$ and $\lim_{x \to 1} g(x)$.

(c) Determine a (potentially piecewise) formula for $h(x) = f(x)g(x)$.

(d) Determine $\lim_{x \to 1} h(x)$. Does this agree with the limit properties above? Explain.

Exercise 1.3.4. Consider the functions f and g below.

$$f(x) = \begin{cases} 2x & \text{if } x > 1 \\ 0 & \text{if } x \leq 1 \end{cases}$$

$$g(x) = \begin{cases} -x-1 & \text{if } x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}$$

(a) Sketch graphs of f and g.

(b) Determine the limits $\lim_{x\to 1} f(x)$ and $\lim_{x\to 1} g(x)$.

(c) Determine a (potentially piecewise) formula for $h(x) = f(x)g(x)$.

(d) Determine $\lim_{x\to 1} h(x)$. Does this agree with the limit properties above? Explain.

Exercise 1.3.5. Create a function $f(x)$ so that $\lim_{x\to 3} f(x)$ does not exist, but $\lim_{x\to 3} |f(x)|$ does exist. Sketch graphs of both f and $|f|$ to illustrate your point.

Exercise 1.3.6. Create functions $f(x)$ and $g(x)$ so that $\lim_{x\to -2} f(x)$ exists, $\lim_{x\to -2} g(x)$ does not exist, but $\lim_{x\to -2} f(x)g(x)$ does exist. Sketch graphs of f, g and fg to illustrate your point.

Exercise 1.3.7. Sketch the graph of a function $f(x)$ which satisfies all of the following conditions.

- $f(3) = 0$
- $f(x) < 0$ for $x < 3$
- $\lim_{x \to 3} f(x)$ does not exist.
- $\lim_{x \to -5} f(x) = -1$.
- $f(-5)$ does not exist.
- $f(x) > 0$ for $x > 3$

Exercise 1.3.8. Explicitly define a function f which satisfies all of the above conditions.

1.4 Continuity

This worksheet discusses material corresponding roughly to section 1.3 of your textbook.

One-Sided Limits

When considering limits of the form $\lim_{x \to a} f(x)$, we are analyzing the behavior of the values of f when the inputs are close to a. Often, the behavior is different depending on whether we are considering inputs less than a or if we are considering inputs greater than a. These considerations are settled by **one-sided limits**.

We write

$$\lim_{x \to a^-} f(x) = L$$

to mean that $f(x)$ can be made as close to L as we like by taking x to be sufficiently close to a, but less than a. This limit is referred to as the **limit from the left**. (The $-$ in the notation indicates that x is to be taken below a.)

Similarly, we write

$$\lim_{x \to a^+} f(x) = L$$

to mean that $f(x)$ can be made as close to L as we like by taking x to be sufficiently close to a, but greater than a. This limit is referred to as the **limit from the right**. (The $+$ in the notation indicates that x is to be taken above a.)

Of course, just like with ordinary limits, these one-sided limits need not exist.

Exercise 1.4.1. Let f be the function below.

$$f(x) = \begin{cases} 1 & \text{if } x < 2 \\ 3x - 4 & \text{if } 2 \le x \le 4 \\ 8 & \text{if } x > 4 \end{cases}$$

Determine the following limits.

(a) $\lim_{x \to 2^+} f(x)$

(b) $\lim_{x \to 2^-} f(x)$

(c) $\lim_{x \to 3^+} f(x)$

(d) $\lim_{x \to 3^-} f(x)$

(e) $\lim_{x \to 4^+} f(x)$

(f) $\lim_{x \to 4^-} f(x)$

Exercise 1.4.2. Draw the graph of a function f for which $\lim_{x \to -1} f(x)$ does not exist, $f(-1)$ is undefined, but both $\lim_{x \to -1^+} f(x)$ and $\lim_{x \to -1^-} f(x)$ exist.

Exercise 1.4.3. Draw the graph of a function f for which $\lim_{x \to 0} f(x)$ does not exist, $f(0)$ is defined, $\lim_{x \to 0^+} f(x)$ does not exist, and $\lim_{x \to 0^-} f(x)$ does exist.

Exercise 1.4.4. Draw the graph of a function f for which $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist and are equal. Does $\lim_{x \to a} f(x)$ exists for your function? Explain.

Continuity

Recall that a function f is **continuous at** a if a is in the domain of f and $\lim_{x \to a} f(x) = f(a)$. It is possible to restate this definition using one-sided limits, as well.

Exercise 1.4.5. Use one-sided limits to restate the definition of continuity. (You may want to refer to your work on Exercise 1.4.4.)

If f is continuous at each point in its domain, then we say that f is **continuous**. If b is a point in the domain of f, and f is not continuous at b, then we say f is **discontinuous** at b.

At this point, it's useful to make the following observations.

If f and g are continuous, then so are $f+g$, $f-g$, fg, f/g (provided $g(a) \neq 0$), and $f \circ g$.

That is, combinations of continuous functions are continuous. So when computing limits of these functions, we can proceed by simply "plugging in."

Exercise 1.4.6. Show that constant functions are continuous.

Exercise 1.4.7. Show that the function $f(x) = x$ is continuous.

Exercise 1.4.8. Use the above results to explain why any polynomial function $p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ is continuous. (Here the c_i's are constants.)

Exercise 1.4.9. Explain why any rational function is continuous. (Recall that f is a rational function if $f = p/q$ where both p and q are polynomial functions.)

Exercise 1.4.10. Let f be the function below.

$$f(x) = \begin{cases} 1 & \text{if } x < 2 \\ 3x - 4 & \text{if } 2 \leq x < 4 \\ 8 & \text{if } x > 4 \end{cases}$$

Determine all points of discontinuity of f.

Exercise 1.4.11. Draw the graph of a function g which is continuous on its domain, $g(0)$ does not exist, but $\lim_{x \to 0} g(x)$ does exist. Explicitly define such a function.

Exercise 1.4.12. Let $h(x) = \dfrac{x^2 + 9x + 14}{x^2 - x - 6}$. Determine $\lim_{x \to 2} h(x)$. Is h continuous at 2? Explain.

Exercise 1.4.13. Let $j(t) = \dfrac{\sqrt{3+t} - \sqrt{3}}{t}$. Determine $\lim_{t \to 0} j(t)$. Is h continuous at 0? Explain.

1.5 Limits involving Infinity

This worksheet discusses material corresponding roughly to section 1.3.3 of your textbook.

Exercise 1.5.1. Consider the function $f(x) = \dfrac{1}{x}$. We'll explore $\lim_{x \to 0^+} f(x)$.

(a) Select 5 positive real numbers which get successively closer to 0.

(b) Evaluate f at each of your chosen real numbers.

(c) Take the smallest of the numbers you've chosen and divide it by 100. Evaluate f at this new number.

(d) What seems to be happening to the values of f as the input gets closer to zero?

(e) How close to 0 does x need to be so that $f(x) > 500$?

(f) Can you make $f(x) > 500000$? How?

(g) Suppose you wanted $f(x) > M$ for some $M > 0$. How close to 0 does x need to be to make this happen? Does this work for any $M > 0$?

The behavior above suggests that as x approaches 0 from above, the value of $f(x) = \frac{1}{x}$ grows without bound. We make the following formal definition. We write $\lim_{x \to a} f(x) = \infty$ to mean that for every $M > 0$ there is a $\delta > 0$ so that whenever $0 < |x - a| < \delta$, $f(x) > M$. Note that in particular f does not need to be defined at a, only near a. Also important to note is that if $\lim_{x \to a} f(x) = \infty$, then according to the definition of a limit, $\lim_{x \to a} f(x)$ does not exist because the function values are not approaching any particular real number.

We also have similar definitions for the one-sided limits $\lim_{x \to a^-} f(x) = \infty$ and $\lim_{x \to a^+} f(x) = \infty$, and also the limits $\lim_{x \to a} f(x) = -\infty$, $\lim_{x \to a^-} f(x) = -\infty$, and $\lim_{x \to a^+} f(x) = -\infty$. Also, notice that if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \pm\infty$, then $\lim_{x \to a} f(x) = \pm\infty$, and if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

Exercise 1.5.2. Use your work above to determine $\lim_{x \to 0^+} \frac{1}{x}$.

Exercise 1.5.3. Determine $\lim_{x \to 0^-} \frac{1}{x}$.

Exercise 1.5.4. Determine $\lim_{x \to 0} \frac{1}{x}$.

Exercise 1.5.5. Determine $\lim_{x \to 0} \frac{1}{x^2}$.

Exercise 1.5.6. Determine $\lim_{x \to \frac{\pi}{2}^-} \tan(x)$.

Exercise 1.5.7. Determine $\lim_{x \to \frac{\pi}{2}^+} \tan(x)$.

Exercise 1.5.8. Determine $\lim_{x \to \frac{\pi}{2}} \tan(x)$.

In general, you can get a sense of a limit using the principle that $\dfrac{1}{\text{big}} = \text{small}$, and $\dfrac{1}{\text{small}} = \text{big}$.

Exercise 1.5.9. Consider the function $f(x) = \dfrac{(x+2)(x-3)}{(2x+1)(x-1)}$.

(a) If x is close to 1, but greater than 1, determine which of the following are positive and which are negative. Use $\oplus$ to denote the positive values and $\ominus$ to denote the negative ones.

- (i) $x + 2$
- (ii) $x - 3$
- (iii) $2x + 1$
- (iv) $x - 1$

(b) If x is close to 1, but greater than 1, determine which of the above are "small" (i.e. close to zero).

(c) If x is close to 1, but greater than 1, is $f(x)$ positive or negative? Is the value large (i.e. much, much bigger than zero), small (close to zero), or neither?

(d) What does this tell you about $\lim_{x\to 1^+} f(x)$?

(e) Use a similar approach to determine $\lim_{x\to 1^-} f(x)$, $\lim_{x\to -\frac{1}{2}^-} f(x)$, and $\lim_{x\to -\frac{1}{2}^+} f(x)$.

We also have the following limits to consider. We write $\lim_{x\to\infty} f(x) = L$ to mean that for any $\epsilon > 0$ there is an $M > 0$ so that if $x > M$ then $|f(x) - L| < \epsilon$. That is, if we can make the function value of f as close to L as we like by taking x to be sufficiently large, then we say the limit of f as x goes to infinity is L. We have a similar definition for $\lim_{x\to-\infty} f(x) = L$.

The next two exercises illustrate one key principal in computing limits at infinity.

Exercise 1.5.10. Determine $\lim_{x\to\infty} \frac{1}{x}$.

Exercise 1.5.11. Determine $\lim_{x\to-\infty} \frac{1}{x}$.

Exercise 1.5.12. Consider again the function $f(x) = \dfrac{(x+2)(x-3)}{(2x+1)(x-1)}$. We'll determine $\lim_{x\to\infty} f(x)$.

(a) Distribute in the numerator and the denominator.

(b) Identify the degree of the polynomial in the numerator and the degree of the polynomial in the denominator.

(c) Call the larger of the two degrees n. If they are the same degree, then that's your n.

(d) Multiply numerator and denominator by $\dfrac{1}{x^n}$ where n is the value you just found.

(e) Distribute in the numerator and the denominator, and simplify.

(f) Take $\lim_{x\to\infty}$ of the expression you found above.

(g) Determine $\lim_{x\to\infty} f(x)$.

Exercise 1.5.13. Determine $\lim_{x\to\infty} \dfrac{x^3 + 4x - 2}{3x^4 - 100x^2 + 1}$.

Exercise 1.5.14. Determine $\lim_{x\to\infty} \dfrac{3x^4 - 100x^2 + 1}{x^3 + 4x - 2}$.

Chapter 2

Derivatives

2.1 Derivatives

This worksheet discusses material corresponding roughly to section 1.4 of your textbook.

Recall that the **derivative of** f **at** a is defined to be

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The derivative is the slope of the tangent line to the graph of f at the point $(a, f(a))$, and also the instantaneous rate of change of f when $x = a$.

Exercise 2.1.1. Let $p(t) = t^3$ denote the position in meters of a moving particle at time t seconds. Determine the velocity of the particle at time 3 seconds. (Recall that velocity is the instantaneous rate of change of position with respect to time.)

Exercise 2.1.2. Find the slope of the tangent line to the function $g(q) = 3q^2 - q + 1$ at the point $(-1, 5)$.

Since the derivative of a function at a point is defined by a limit, it may or may not exist. If $f'(a)$ exists, we say that f is **differentiable** at a.

Exercise 2.1.3. Let f be the function defined below.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ x & \text{if } x > 1 \end{cases}$$

(a) Show that f is continuous at 1.

(b) Determine whether or not f is differentiable at 1.

(c) Sketch a graph of f.

Exercise 2.1.4. Let g be the function below.

$$g(x) = \begin{cases} 2x^2 - x + 1 & \text{if } x \leq 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$$

(a) Show that g is continuous at 0.

(b) Determine whether or not g is differentiable at 0.

(c) Sketch a graph of g.

Exercise 2.1.5. Determine all points at which the function $f(x) = |x|$ is differentiable. It may help to recall that $|x|$ can be defined piecewise as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Exercise 2.1.6. Determine whether or not the function $f(x) = x^{1/3}$ is differentiable at 0. Illustrate with a graph.

Exercise 2.1.7. Determine whether or not the function $g(x) = x^{2/3}$ is differentiable at 0. Illustrate with a graph.

2.2 Extrema

This worksheet discusses material corresponding roughly to section 1.5 of your textbook.

The main goal of these exercises is to understand and make use of the following theorem

Theorem. If f attains a local extreme value at a, then a is a critical point of f.

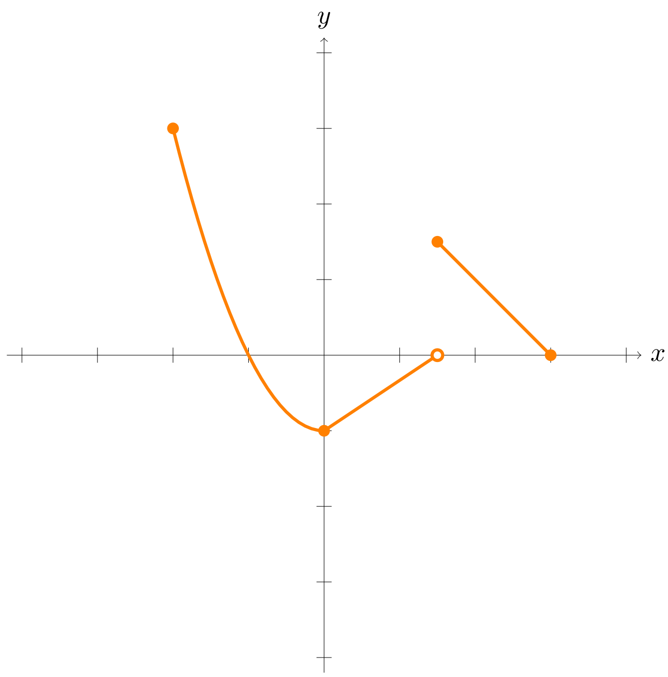
To get there, we need some terminology.

Definition 2.2.1. A function f **attains a global maximum value of** M **at a point** a **in the domain of** f if $f(a) = M$ and $f(x) \leq M$ for every point x in the domain of f.

Similarly, f **attains a global minimum value of** m **at a point** a **in the domain of** f if $f(a) = m$ and $f(x) \geq m$ for every point x in the domain of f.

The global maximum value and the global minimum value are collectively referred to as the **global extreme values of** f.

Exercise 2.2.2. The graph of a function f is given below. Identify all points in the domain of f at which f attains a global maximum value and all points at which f attains a global minimum value. Also determine these global extreme values. (Assume each tick displayed is 1 unit.)



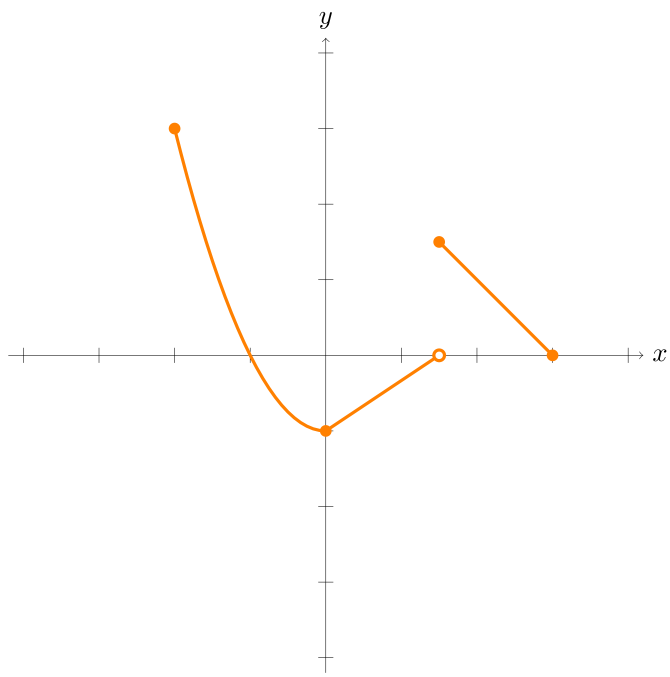
We similarly have the following definitions.

Definition 2.2.3. A function f **attains a local maximum value of** M **at a point** a **in the domain of** f if $f(a) = M$ and $f(x) \leq M$ for every point x in some (small) open interval around a.

Similarly, f **attains a local minimum value of** m **at a point** a **in the domain of** f if $f(a) = m$ and $f(x) \geq m$ for every point x in some (small) open interval around a.

The local maximum values and the local minimum values are collectively referred to as the **local extreme values of** f.

Exercise 2.2.4. The graph of a function f is given below. Identify all points in the domain of f at which f attains a local maximum value and all points at which f attains a local minimum value. Also determine these local extreme values. (Assume each tick displayed is 1 unit.)

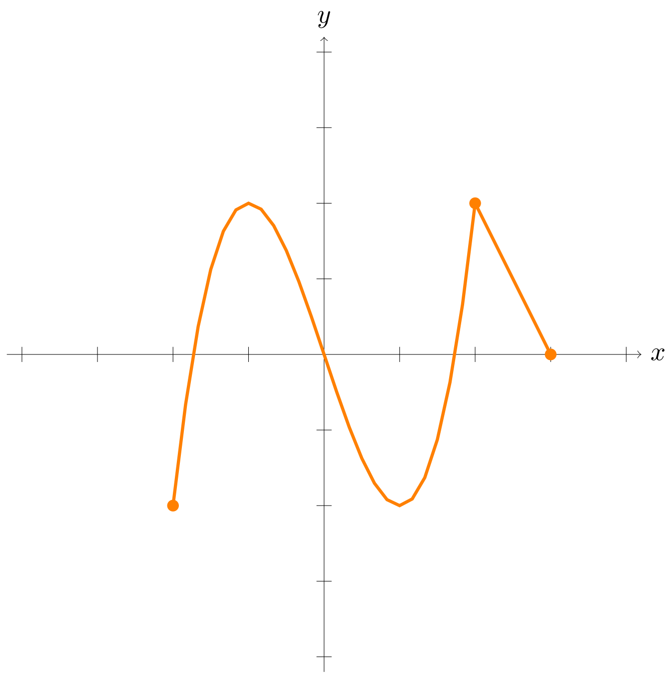


Exercise 2.2.5. Consider the function $f(x) = x^2 - 1$ with domain $(-\infty, 1]$. Sketch the graph of f and use it to estimate all local and global extreme values.

More definitions.

Definition 2.2.6. A **critical point** of a function f is a point a in the domain of f at which either $f'(a) = 0$ or at which f is not differentiable. A **critical value** of f is the value of f at a critical point.

Exercise 2.2.7. Consider the graph of the function g depicted below. Determine all critical points and critical values of g. (Be careful. Is g differentiable at the endpoints of its domain?)



Exercise 2.2.8. Determine all local extreme values of the function g above. Which of these local extreme values occur at critical points?

The following theorem explains the phenomenon you observed in the previous exercise.

Theorem. If f attains a local extreme value at a, then a is a critical point of f.

Exercise 2.2.9. Let $f(x) = (x-2)(x-1)(x+1)$ with domain $[-3, 4]$. Determine all critical points of f.

If f is a function and $f'(x) > 0$ at every point x in the interval (a, b), then we say that f is **increasing** on (a, b). Similarly, if f is a function and $f'(x) < 0$ at every point x in the interval (a, b), then we say that f is **decreasing** on (a, b).

Exercise 2.2.10. Let f be the function from the previous exercise. Determine all intervals on which f is increasing and all intervals on which f is decreasing.

2.3 Basic Differentiation Rules

This worksheet discusses material corresponding roughly to section 2.1 of your textbook.

We want to compute derivatives because they tell us about the rate of change of a function, but each derivative we compute involves a limit, which can be difficult and time consuming to evaluate. Fortunately, we can make use of some patterns which arise and create rules for differentiation based on these patterns. The first such rule is the so-called **power rule**.

Theorem (Power Rule)**.** Suppose that n is a positive integer. Then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

For example, $\frac{d}{dx}(x^2) = 2x$ and $\frac{d}{dt}(t^7) = 7t^6$. Also, recall that $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(c) = 0$ if c is a constant.

Exercise 2.3.1. Compute the derivatives of the following functions.

(a) $a(k) = k^5$

(b) $b(\ell) = \ell^{10}$

(c) $c(m) = m^4$

(d) $d(n) = n^{21}$

(e) $e(o) = o^{100}$

(f) $f(p) = p^{29561}$

(g) $g(q) = q$

(h) $h(r) = r^{11}$

(i) $i(s) = s^{51}$

(j) $j(t) = t^{99}$

(k) $k(u) = u^{100}$

(l) $\ell(v) = v^2$

(m) $m(w) = w^{456}$

(n) $n(x) = x^{56}$

Then next rule is **linearity**.

Theorem (Linearity)**.** Suppose f and g are differentiable at x and $a, b \in \mathbb{R}$ are constants. Then $af + bg$ is differentiable at x and

$$(af + bg)'(x) = af'(x) + bg'(x)$$

For example, $\frac{d}{dx}(4x^3 - 2x^7) = 4\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^7) = 4(3x^2) - 2(7x^6) = 12x^2 - 14x^6$.

Exercise 2.3.2. Compute the derivatives of the following functions.

(a) $f(x) = 2x^8 + 3x^{10}$

(b) $g(y) = -5y + 13$

(c) $h(z) = \frac{1}{2}z^3 - \frac{4}{5}z^5 + 12z$

(d) $k(a) = \sqrt{3}a^7 - \pi a^2$

(e) $\ell(b) = -\frac{\sqrt{2}}{2} + b - \frac{17}{19}b^{20} + 10238461917395$

(f) $m(c) = 1 + c + c^2 + c^3 + c^4 + c^5$

(g) $n(d) = 1 - \frac{d^2}{2} + \frac{d^4}{24} - \frac{d^6}{720}$

(h) $o(e) = \frac{1 - e + e^2}{24}$

(i) $p(f) = \frac{f^2 - f}{f}$

(j) $q(g) = (1 + g)^2$

(k) $r(h) = 2(1 - r)^3$

Exercise 2.3.3. Let $f(x) = x^3 + 4x^2 - 3x - 4$.

(a) Determine $f'(x)$.

(b) Determine all critical points of f.

(c) Determine all intervals of increase/decrease of f.

(d) Classify all critical points of f as either local maxima, local minima, or neither.

Exercise 2.3.4. Suppose $f'(3) = -2$ and $g'(3) = 3$. Determine $(3f - 2g)'(3)$.

Exercise 2.3.5. A particle is moving along a straight line with position given by $x(t) = 3t^5 - 10t^3$ meters at time t seconds.

(a) Determine all intervals on which the velocity of the particle is increasing and those on which the velocity is decreasing.

(b) On what intervals is the particle accelerating? On what intervals is the particle decelerating?

(c) Determine the maximum and minimum values of the acceleration of the particle on the time interval $-2 \leq t \leq 1$.

2.4 Product Rule, Quotient Rule, and Chain Rule

This worksheet discusses material corresponding roughly to sections 2.2–2.3 of your textbook.

Recall the following theorem.

Theorem. Suppose that f and g are differentiable at x.

- (Product Rule) Then fg is differentiable at x and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

- (Quotient Rule:) If $g(x) \neq 0$, then f/g is differentiable at x and

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Start with just some practice.

Exercise 2.4.1. Compute the following derivatives.

(a) $\dfrac{d}{dt}\left((3t+5)(t^2-2t-1)\right)$

(b) $\dfrac{d}{dr}\left(\dfrac{3r-4}{2r-1}\right)$

(c) $\dfrac{d}{dz}\left((z+1)^3\right)$

Exercise 2.4.2. Create functions f and g so that neither f nor g is differentiable at 3, but fg is differentiable at 3.

Exercise 2.4.3. Suppose $f(2) = 1$, $f'(2) = -3$, $g(2) = 4$, and $g'(2) = 0$.

(a) Determine $(fg)'(2)$.

(b) Determine $\left(\frac{f}{g}\right)'(2)$.

(c) Determine $(f^2 - 3g + 4)'(2)$.

We also have the following rule for differentiating the composition of two functions.

Theorem (Chain Rule)**.** Suppose that g is differentiable at x and f is differentiable at $g(x)$. Then $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Exercise 2.4.4. Let $f(x) = x^7$ and $g(x) = 4x^2 + 1$.

(a) Compute $f'(x)$.

(b) Compute $g'(x)$.

(c) Compute the composition $(f \circ g)(x) = f(g(x))$.

(d) Compute $(f \circ g)'(x)$.

Exercise 2.4.5. Let $h(t) = (7t^3 + 4t - 1)^3$. Compute $h'(t)$.

Exercise 2.4.6. Let $j(q) = (q^2 + 1)^2(2q - 1)$. Compute $j'(q)$.

Exercise 2.4.7. Let $x(f) = (3f-1)^7(6f+1)^9$. Compute $x'(f)$.

Exercise 2.4.8. Let $f(x) = \dfrac{(3x+1)(2x-9)}{x+1}$. Compute $f'(x)$.

Exercise 2.4.9. Use the quotient rule to compute the derivatives below.

(a) Compute $\frac{d}{dx}\left(\frac{1}{x}\right)$.

(b) Compute $\frac{d}{dx}\left(\frac{1}{x^2}\right)$.

(c) Compute $\frac{d}{dx}\left(\frac{1}{x^3}\right)$.

(d) Compute $\frac{d}{dx}\left(\frac{1}{x^4}\right)$.

(e) Generalize.

2.5 The Exponential Function

This worksheet discusses material corresponding roughly to sections 2.4 of your textbook.

We start by defining a function exp by stating the properties we wish this function to have. The function $\exp\colon \mathbb{R} \to \mathbb{R}$ is defined by

- $\exp(0) = 1$
- $\dfrac{d}{dx}(\exp(x)) = \exp(x)$.

Natural questions at this point are "Why does such a function exist?", "Why do these properties uniquely define exp?", and "Isn't $\exp(x)$ just the same thing as e^x?" These are fair questions and will be addressed. (Eventually.) Also, note that the *name* of the function is exp. So, in particular, $\exp \neq \text{pex} \neq \text{epx} \neq \text{xep}$, etc.

Exercise 2.5.1. Compute the following derivatives. Don't forget to apply the Chain Rule, Product Rule, and Quotient Rule when appropriate.

(a) $\dfrac{d}{dx}\left(\exp(x) + x^2 - 1\right)$

(b) $\dfrac{d}{dx}\left(4\exp(x) - 3x^{10}\right)$

(c) $\dfrac{d}{dx}\left(\exp(3x)\right)$

(d) $\dfrac{d}{dx}\left(x\exp(x)\right)$

(e) $\dfrac{d}{dx}\left(\dfrac{x^2+1}{\exp(x)}\right)$

Exercise 2.5.2. Consider the function $f(x) = \exp(x)\exp(-x)$.

(a) Show that $f'(x) = 0$.

(b) Conclude that $f(x)$ is a constant function, $f(x) = C$.

(c) Determine $f(0)$.

(d) Conclude that $\exp(x)\exp(-x) = 1$ for all x.

(e) Now show that $\exp(-x) = \frac{1}{\exp(x)}$.

(f) Is $\exp(x)$ ever equal to zero? Explain.

(g) Is $\exp(x)$ ever negative? Explain.

(h) Is $\exp(x)$ always positive? Explain.

(i) On what intervals is exp increasing? Explain.

Exercise 2.5.3. Suppose g is a function so that $g'(x) = kg(x)$ for some constant k.

(a) Show that $f(x) = \exp(kx)$ is such a function. That is, $f'(x) = kf(x)$.

(b) Show that the derivative $\frac{d}{dx}\left(\frac{g(x)}{\exp(kx)}\right)$ is 0.

(c) Explain why this means that $\frac{g(x)}{\exp(kx)} = C$ for some constant C.

(d) Explain why there is exactly one function g which satisfies $g'(x) = g(x)$ and $g(0) = 1$. What function is this?

Exercise 2.5.4. Show that $\exp(x+a) = \exp(x)\exp(a)$ for any real numbers x and a. (*Hint*: Divide and use the technique from the previous exercises.)

Exercise 2.5.5. Show that $\exp(rx) = (\exp(x))^r$ for any rational number r.

2.6 The Natural Logarithm

This worksheet discusses material corresponding roughly to sections 2.5 - 2.6 of your textbook.

Recall that exp is the function $\exp\colon \mathbb{R} \to (0, \infty)$ is defined by

- $\exp(0) = 1$
- $\dfrac{d}{dx}(\exp(x)) = \exp(x)$.

Also recall that the following properties hold for the exponential function exp:

- $\exp(a + x) = \exp(a)\exp(x)$
- $\exp(rx) = (\exp(x))^r$
- exp is always increasing and always concave up
- $\lim_{x\to-\infty} \exp(x) = 0$
- $\lim_{x\to\infty} \exp(x) = \infty$
- $\exp(x) > 0$ for all x
- $e = \exp(1) \approx 2.71828182846$
- $\exp(x) = e^x$ for all x

Exercise 2.6.1. Use all of the above information (or at least some of the above information) to sketch a graph of exp.

Exercise 2.6.2. Does the graph you drew above pass the "horizontal line test"? That is, does every horizontal line intersect the graph of exp in at most 1 point?

Since exp passes the horizontal line test, it's invertible. So exp has an inverse function, which we will denote ln and refer to as the **natural logarithm**. Explicitly, ln is the function which satisfies the following properties.

- $\ln\colon (0,\infty) \to \mathbb{R}$
- $\ln(\exp(x)) = x$ for all real numbers x
- $\exp(\ln(y)) = y$ for all $y > 0$
- $\ln(1) = 0$, and $\ln(e) = 1$
- $\lim_{x\to\infty} \ln(x) = \infty$, and $\lim_{x\to 0^+} \ln(x) = -\infty$

Now you'll compute the derivative of ln.

Exercise 2.6.3. Let $y = \ln(x)$.

(a) Solve the equation for x.

(b) Differentiate each side of the equation you just produced with respect to x. That is, take $\frac{d}{dx}$ of each side. (*Warning*: y is a function of x, so you will need to use the Chain Rule.)

(c) Solve the previous equation for y' (or $\frac{dy}{dx}$).

(d) Plug in $y = \ln(x)$ to the equation for y'.

(e) Conclude that $\frac{d}{dx}\ln(x) = \frac{1}{x}$.

Exercise 2.6.4. Compute the derivatives below. Don't forget to apply the Chain Rule, Product Rule, and Quotient Rule when appropriate.

(a) $\dfrac{d}{dx}\big(\ln(x) - x^2 + \exp(x)\big)$

(b) $\dfrac{d}{dx}\big(\ln(x^2 + 2x + 1)\big)$

(c) $\dfrac{d}{dx}\left(x \ln(x)\right)$

(d) $\dfrac{d}{dx}\left(\ln(\exp(x) + 1)\right)$

Exercise 2.6.5. Show that ln has each of the following properties.

(a) If $x > 0$, then $\ln(1/x) = -\ln(x)$.

(b) For all $x > 0$ and all $a > 0$, $\ln(ax) = \ln(a) + \ln(x)$.

(c) For all $x > 0$ and all $a > 0$, $\ln(a/x) = \ln(a) - \ln(x)$.

(d) For all $x > 0$ and all rational numbers r, $\ln(x^r) = r\ln(x)$.

2.7 Trigonometric Functions

This worksheet discusses material corresponding roughly to sections 2.7 - 2.8 of your textbook.

Recall from class the following limits:

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1$$

$$\lim_{h\to 0} \frac{1-\cos(h)}{h} = 0$$

Exercise 2.7.1. Use the limit definition of the derivative to compute $\frac{d}{dt}\cos(t)$. Recall that you will likely need to use a trig identity to work this out.

Also, recall from class that $\frac{d}{dt}\sin(t) = \cos(t)$.

Exercise 2.7.2. Compute $\frac{d^n}{dt^n}\sin(t)$ for all non-negative integers n. That is, compute all derivatives of $\sin(t)$ of all orders. Then do the same for $\cos(t)$.

Recall that the remaining standard trigonometric functions are defined in terms of sine and cosine as follows:

$$\begin{aligned}
\tan(t) &= \frac{\sin(t)}{\cos(t)} \\
\cot(t) &= \frac{\cos(t)}{\sin(t)} \\
\sec(t) &= \frac{1}{\cos(t)} \\
\csc(t) &= \frac{1}{\sin(t)}
\end{aligned}$$

Exercise 2.7.3. Compute the following derivatives. In each case, write the result in terms of tan, cot, sec, and csc.

(a) $\dfrac{d}{dt}\tan(t)$

(b) $\dfrac{d}{dt}\cot(t)$

(c) $\dfrac{d}{dt}\sec(t)$

(d) $\dfrac{d}{dt}\csc(t)$

Exercise 2.7.4. Compute the derivatives of the functions below.

(a) $f(x) = x\sin(x)$

(b) $g(y) = \sec(3y)$

(c) $h(z) = \sec(z)\tan(z)$

(d) $i(a) = \exp(a)\cos(a)$

(e) $j(b) = \dfrac{\sec(b)}{\csc(b)}$

(f) $k(c) = \sin(\sin(c))$

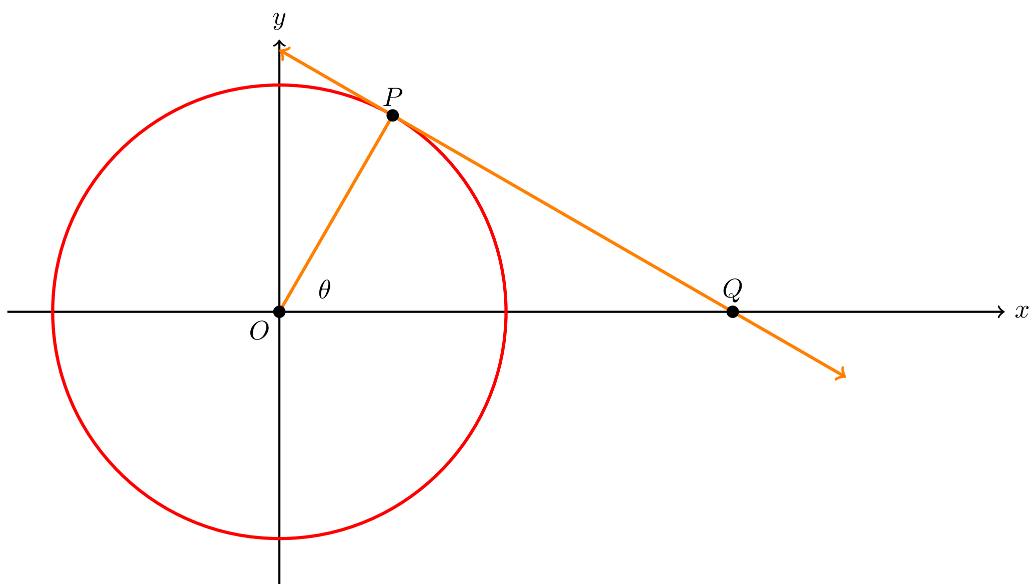
(g) $\ell(d) = \exp(\tan(-2d))$

(h) $m(f) = \ln(\tan(f))$

(i) $n(g) = \sec^2(g) - \tan^2(g)$

Exercise 2.7.5. Determine all critical points of the function $f(x) = e^{-x}\sec(x)$. Classify them.

Exercise 2.7.6. Consider the diagram below.



(a) Let s be the length of the segment $\overline{PQ}$ and t be the length of the segment $\overline{OQ}$. Express both s and t in terms of θ.

(b) Compute $\dfrac{ds}{d\theta}$ and $\dfrac{dt}{d\theta}$.

(c) Determine $\lim_{\theta \to 0^+} \dfrac{ds}{d\theta}$, $\lim_{\theta \to 0^+} \dfrac{dt}{d\theta}$, $\lim_{\theta \to \frac{\pi}{2}^-} \dfrac{ds}{d\theta}$, and $\lim_{\theta \to \frac{\pi}{2}^-} \dfrac{dt}{d\theta}$.

2.8 Inverse Trigonometric Functions

This worksheet discusses material corresponding roughly to section 2.9 of your textbook.

Recall from class that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$. We will use a similar method to determine the derivatives of the inverse sine and inverse cosine functions. Recall that $y = \arcsin(x)$ means that $\sin(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, and that $y = \arccos(x)$ means that $\cos(y) = x$ and $0 \leq y \leq \pi$.

Exercise 2.8.1. Let $y = \arcsin(x)$.

(a) Solve the equation for x.

(b) Take $\frac{d}{dx}$ of the equation. (Remember, y is a function of x, so you will need to use the chain rule.)

(c) Solve for y'.

(d) Replace y with $\arcsin(x)$ (since $y = \arcsin(x)$).

(e) Use a triangle to resolve the composition of cos with arcsin. (You should get something involving 1's, x's, and $\sqrt{\ }$'s.)

(f) What is $\frac{d}{dx}\arcsin(x)$?

Exercise 2.8.2. Determine $\dfrac{d}{dx}\arccos(x)$.

Exercise 2.8.3. Compute the derivatives below.

(a) $\dfrac{d}{dx}\left(\arctan(3x+1)\right)$

(b) $\dfrac{d}{dx}\left(x\arcsin(x)\right)$

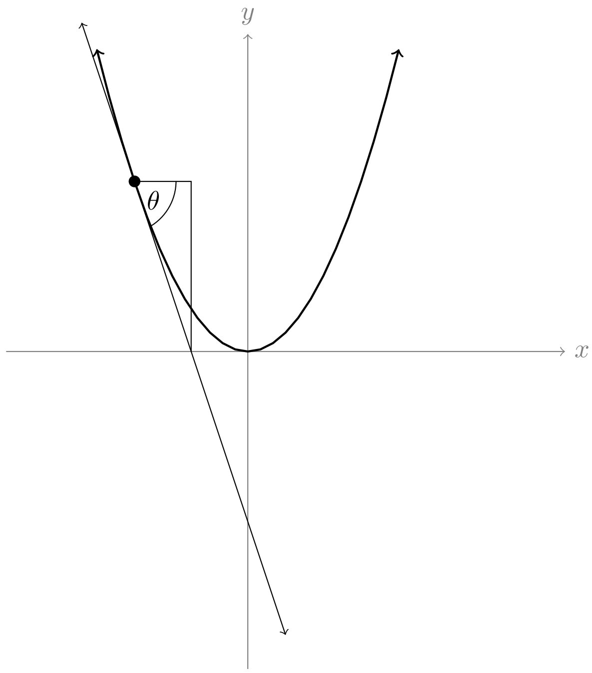
(c) $\dfrac{d}{dx}\left(\arccos(\exp(x))\right)$

(d) $\dfrac{d}{dx}\left(x\arctan(x)-\dfrac{1}{2}\ln(1+x^2)\right)$

(e) $\dfrac{d^2}{dx^2}\left(\arctan(x)\right)$

(f) $\dfrac{d^2}{dx^2}\left(\arcsin(x)\right)$

Exercise 2.8.4. Consider the diagram below.



The angle labeled as θ is the **angle of declination** of the curve. That is, θ is the angle that the tangent line to the curve makes with the horizontal. In this case, the curve is the familiar $y = x^2$.

(a) Determine an expression for θ in terms of x.

(b) Determine $\frac{d\theta}{dx}$ when $x = -1$.

Chapter 3

Applications of Derivatives

3.1 Related Rates

This worksheet discusses material corresponding roughly to section 3.1 of your textbook.

Related rates problems involve relationships between quantities which are themselves functions of some other parameter. This other parameter is often time, but doesn't need to be.

Exercise 3.1.1. Recall that the volume of a cylinder is given by $V = \pi r^2 h$, where V is the volume, r is the radius of the base, and h is the height of the cylinder.

(a) Write an equation for the volume of a cylinder with height 1 cm.

(b) Suppose that both V and r are functions of t. Compute $\frac{d}{dt}$ of each side of the equation from the previous part.

(c) If $\frac{dr}{dt} = 2$ cm/min and $r = 4$ cm, then what is $\frac{dV}{dt}$?

Exercise 3.1.2.

Suppose a the height of a cylinder is increasing at a rate of 4 cm/s and the radius is decreasing at a rate of 2 cm/s. At what rate is the volume changing when the height is 20 cm and the radius is 10 cm? (Note: Now each of h, r, and V are functions of time t.)

Exercise 3.1.3. A spherical balloon is being inflated so that the volume increases at a rate of 2 cm^3 per second. At what rate is the radius changing when the radius is 10 cm?

Exercise 3.1.4. Sand falls in an hourglass at a constant rate of 2 cubic inches every 3 minutes, creating a conical pile of sand in the lower bulb of the hourglass. The sand falls in such a way that the height of the pile is always half of the radius of the pile. At what rate is the height of the pile increasing when the height is 1 inch?

Exercise 3.1.5. A lighthouse is on an island 2 miles off of a long, straight shore. The light makes 2 revolutions per minute, and shines a beam of light onto the shore making a spot of light just where the shore meets the ocean. How fast is the spot of light on the shore moving when the spot of light is 4 miles from the point on the shore nearest the lighthouse?

Exercise 3.1.6. A 6 foot tall woman is walking away from a light post which is 20 feet high. The light casts a shadow of the woman on the ground in front of her. How is the height of her shadow changing when she is 40 feet from the light post and walking at a rate of 4 feet per second?

3.2 Graphing & Optimization

This worksheet discusses material corresponding roughly to sections 3.2 – 3.3 of your textbook.

Recall that a function f is **concave up** if $f''(x) > 0$, and so the graph is $\cup$ shaped. Similarly, a function is **concave down** if $f''(x) < 0$, and so the graph is $\cap$ shaped. An **inflection point** is a point in the domain of f where the concavity changes.

Exercise 3.2.1. Consider the function $f(x) = \dfrac{x^2}{x-1}$.

(a) Determine the domain, all roots, asymptotes, intervals of increase/decrease, intervals of concavity, local extrema, and inflection points of f.

(b) Use the information you collected above to sketch a graph of f.

Exercise 3.2.2. Repeat the previous exercise for the function $g(x) = (x + 2)\exp(-2x)$.

Here's an optimization problem.

Exercise 3.2.3. A metal can is needed to hold 30 cubic inches of beet and bacon soup. The can is to be constructed of aluminum and will be cylindrical in shape. The cost of material for the side of the can is \$0.05 per square inch, which the cost for the base and the top is \$0.10 per square inch.

(a) What range of values are possible for the radius and the height of the can?

(b) Write an equation for the volume of the can in terms of the radius and the height.

(c) Write an equation for the cost of the can in terms of the radius and the height.

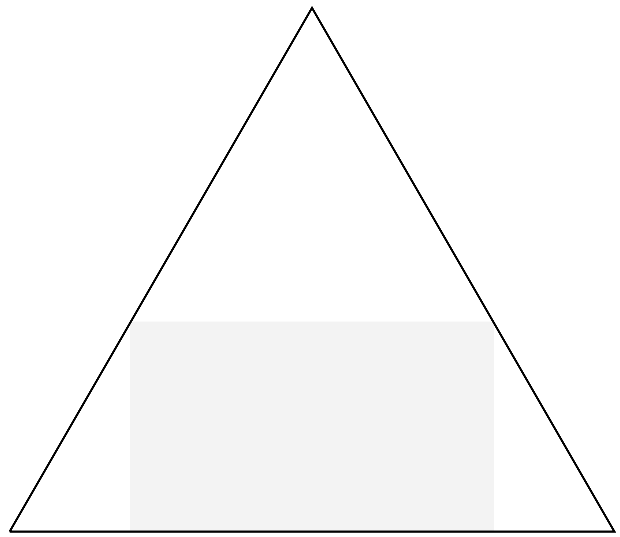
(d) Use the volume equation to solve for either h or r in terms of the other. (i.e. Solve for h in terms of r, or r in terms of h.)

(e) Plug this into the cost equation, so that now cost is a function of a single variable (either h or r).

(f) Use calculus to find the global minimum value of the cost function.

(g) What dimensions of the can (radius and height) will minimize the cost of the can?

Exercise 3.2.4. Find the dimensions of the rectangle inscribed in an equilateral triangle in the fashion illustrated below which has the largest possible area.



3.3 Linear Approximation

This worksheet discusses material corresponding roughly to section 3.4 of your textbook.

Given a function f and a point a in the domain of f, the **linear approximation of f at a** is the function $L_f(x;a)$ defined by

$$L_f(x;a) = f(a) + f'(a)(x-a)$$

Notice this function is the tangent line to f at a.

Exercise 3.3.1. Let $f(x) = \sqrt{1+x}$. Determine $L_f(x;3)$.

The linear approximation of a function at a point is just that: an approximation which is linear. So for values of x near a, $L_f(x;a) \approx f(x)$.

Exercise 3.3.2. Use your linear approximation above to approximate $\sqrt{4.1}$ and $\sqrt{3.9}$. **Don't use a calculator to determine your approximations.** Then use a calculator to see how far off your approximations are.

Of course, when you're in the business of approximating, you want to know that (1) your approximations are "good," and (2) how good your approximation is without knowing the exact value of the thing you're approximating.

The **remainder** when approximating f by $L_f(x;a)$ is the function $R_f(x;a)$ given by

$$R_f(x;a) = f(x) - L_f(x;a)$$

The theorem below tells us that, at least for nice functions f, the remainder can be approximated using the second derivative of f.

Theorem. Suppose that f is twice differentiable on an open interval I around a point a. Then, for all $x \neq a$ in I, there exists a point c between x and a such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$$

i.e. such that $R_f(x;a) = \frac{f''(c)}{2}(x-a)^2$.

So for each x near a, there is a c so that the remainder when approximating f by $L_f(x;a)$ is given by $\frac{f''(c)}{2}(x-a)^2$. The thing is, this c depends on the point x, so it changes. But, if we can get a bound on $\frac{f''(x)}{2}(x-a)^2$ for values of x near a, then we will have a bound on our error.

Exercise 3.3.3. Consider again $f(x) = \sqrt{1+x}$. We will approximate f near $a = 3$ as before, and consider the error when $|x-3| \leq 0.1$, that is for values of x in the range $2.9 \leq x \leq 3.1$.

(a) Compute $f''(x)$.

(b) Check that f'' is an increasing function. (i.e. Take another derivative and see that this is always positive.)

(c) Explain why for any c between 2.9 and 3.1 the following inequalities hold

$$f''(2.9) < f''(c) < f''(3.1) < 0$$

(d) Explain why $|f''(c)| < |f''(2.9)| = \frac{1}{4(3.9)^{3/2}}$.

(e) Explain why $|R_f(x;3)| < \frac{1}{8(3.9)^{3/2}}(0.1)^2$.

Of course, we could compute this last expression with a calculator, but let's make some further estimates without a calculator.

(f) Which is bigger: 3.9 or $(1.5)^2$?

(g) Which is bigger: $\frac{1}{3.9}$ or $\frac{1}{(1.5)^2}$?

(h) Which is bigger: $\frac{1}{8(3.9)^{3/2}}(0.1)^2$ or $\frac{1}{8((1.5)^2)^{3/2}}(0.1)^2$?

(i) **Without a calculator**, simplify $\frac{1}{8(1.5)^3}(0.1)^2$.

(j) Which is bigger: The expression you obtained in the previous part, or $\frac{1}{2500}$.

(k) Compute a decimal expression for $\frac{1}{2500}$ **without a calculator**.

(l) Check that the actual error in your computations from Exercise 2 is smaller than the amount in the previous part.

(m) OK, now use a calculator to check that the error in your computations from Exercise 2 is smaller than $\frac{1}{8(3.9)^{3/2}}(0.1)^2$.

3.4 l'Hôspital's Rule

This worksheet discusses material corresponding roughly to section 3.5 of your textbook.

The following symbols represent **indeterminate forms**.

$$\frac{0}{0} \qquad \frac{\pm\infty}{\pm\infty} \qquad 0\cdot(\pm\infty) \qquad \infty-\infty \qquad 0^0 \qquad 0^\infty \qquad \infty^0 \qquad \infty^\infty \qquad 1^\infty$$

Each of these indeterminate forms is undefined. However, when these arise in computing limits, we often have some ways of working with the expression to resolve the limit.

Exercise 3.4.1. Consider the limit $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

(a) Which indeterminate form above does this limit correspond to?

(b) Evaluate the limit using techniques you know.

Exercise 3.4.2. Consider the limit $\lim_{x\to \infty} \frac{x^2-1}{x-1}$.

(a) Which indeterminate form above does this limit correspond to?

(b) Evaluate the limit using techniques you know.

Exercise 3.4.3. Consider the limit $\lim_{x\to \infty} \frac{x}{\exp(x)}$.

(a) Which indeterminate form above does this limit correspond to?

(b) Observe that neither method you used above applies to this limit.

Here we have a new tool to help us compute limits.

Theorem 3.4.4 (l'Hôspital's Rule)**.** Let $c \in \mathbb{R}$ or $c = \pm\infty$ and suppose that $\lim_{x \to c} \frac{f(x)}{g(x)}$ is indeterminate of the form $\frac{\pm\infty}{\pm\infty}$ or $\frac{0}{0}$. Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the latter limit exists.

(Some technical details have been omitted to simplify the statement. For a complete, precises statement of l'Hôspital's Rule, see page 401 of your text.)

Exercise 3.4.5. Consider the limit $\lim_{x \to \infty} \frac{x}{\exp(x)}$.

(a) Does l'Hôspital's Rule apply to this limit? Explain.

(b) If l'Hôspital's Rule applies, use it to compute the limit.

Exercise 3.4.6. (a) Give an example of a limit for which l'Hôspital's Rule does not apply, but one which you can compute a limit for. Explain why it does not apply.

(b) Compute your limit.

(c) Apply l'Hôspital's Rule to your limit, even though it doesn't apply, and compute the resulting limit. Do you get the same result?

Exercise 3.4.7. Compute the following limits.

(a) $\lim_{x\to\infty} \frac{\ln(x)}{x}$

(b) $\lim_{x\to 0^+} \frac{\ln(x)}{x}$

(c) $\lim_{\theta\to 0} \frac{\theta - \sin(\theta)}{\theta^3}$

(d) $\lim_{t\to 4} \frac{t^2 - 4t + 2}{t^2 - 1}$

Exercise 3.4.8. Create functions f and g so that all of the following hold:

- $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is indeterminate of the form $\frac{\infty}{\infty}$.
- $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ is indeterminate of the form $\frac{\infty}{\infty}$.
- $\lim_{x\to\infty} \frac{f''(x)}{g''(x)}$ is indeterminate of the form $\frac{\infty}{\infty}$.
- $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 12$.

Show that each of the given properties holds for your f and g.

Exercise 3.4.9. Compute: $\lim_{x\to 0^+} x\ln(x)$.

Exercise 3.4.10. Compute: $\lim_{p\to\infty} \left(1+\frac{x}{p}\right)^p$.

Chapter 4

Anti-derivatives

4.1 Differential Equations

This worksheet discusses material corresponding roughly to sections 4.1 - 4.2 of your textbook.

Simply put, a **differential equation** is one which involves derivatives of functions. A function is a **solution** to a differential equation if after "plugging" it in, the resulting equation is true.

Exercise 4.1.1. Show that $y = \exp(2x)$ is a solution to the differential equation $\dfrac{dy}{dx} = 2y$.

Exercise 4.1.2. Determine which functions below are solutions to the differential equation

$$\frac{dy}{dx} = xy$$

(a) $y = x\sin(x)$

(b) $y = 5\exp(x^2/2)$

(c) $y = \ln(x)/x$

(d) $y = -2\exp(x^2/2)$

Exercise 4.1.3. Determine which functions below are solutions to the differential equation

$$\left(\frac{dy}{dx}\right)^2 = 1 - y^2$$

(a) $y = \sin(x)$

(b) $y = \cos(x)$

(c) $y = \sin^2(x)$

(d) $y = -\sin(x - \pi)$

Exercise 4.1.4. Determine which functions below are solutions to the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x^2 - 1$$

(a) $y = 2x^2 - 6x + \dfrac{13}{2}$

(b) $y = \exp(-2x)$

(c) $y = \exp(-x)$

(d) $y = \exp(-2x) + \exp(-x) + 2x^2 - 6x + \dfrac{13}{2}$

For most of what we'll be doing, we'll be interested in solving differential equations of the form

$$\frac{dy}{dx} = f(x)$$

where f is some known function. A solution $y = F(x)$ to the equation $\frac{dy}{dx} = f(x)$ is called an **anti-derivative** of f.

Exercise 4.1.5. Find at least three solutions to each of the equations below.

(a) $\frac{dy}{dx} = 2x$

(b) $\frac{dy}{dx} = \exp(x)$

(c) $\frac{dy}{dx} = \frac{1}{1+x^2}$

(d) $\frac{dy}{dx} = 3x^2 - \sin(x) + \frac{1}{x} + 5$

(e) $\frac{dy}{dx} = \exp(-x) + 1$

As you saw above, anti-derivatives are not unique. In fact, if $F(x)$ is an anti-derivative of $f(x)$, then so is $F(x)+C$ for any constant C. In this case, we call $F(x)+C$ the **general anti-derivative** of f. The notation for the general anti-derivative of f is

$$\int f(x)\,dx = F(x) + C$$

Exercise 4.1.6. Determine the following general anti-derivatives.

(a) $\int 0\,dx$

(b) $\int 1\,dx$

(c) $\int x^p\,dx$, if $p \neq -1$.

(d) $\int x^{-1}\,dx$

(e) $\int \exp(x)\,dx$

(f) $\int \sin(x)\,dx$

(g) $\int \cos(x)\,dx$

(h) $\int \frac{1}{1+x^2}\,dx$

(i) $\int \frac{1}{\sqrt{1-x^2}}\,dx$

(j) $\int \sec(x)\tan(x)\,dx$

4.2 The Substitution Method

This worksheet discusses material corresponding roughly to section 4.2 of your textbook.

Each rule for differentiation has a corresponding rule for anti-differentiation.

Exercise 4.2.1. Rewrite each rule for differentiation below as a rule for anti-differentiation. The first one is done for you.

(a) $\frac{d}{dx}\sin(x) = \cos(x)$ $\qquad\qquad \int \cos(x)\,dx = \sin(x) + C$

(b) $\frac{d}{dx}\exp(x) = \exp(x)$

(c) $\frac{d}{dx}\cos(x) = -\sin(x)$

(d) $\frac{d}{dx}\tan(x) = \sec^2(x)$

(e) $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

(f) $\frac{d}{dx}\ln|x| = \frac{1}{x}$

(g) $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

(h) $\frac{d}{dx}af(x) = af'(x)$

Exercise 4.2.2. The chain rule leads to a technique of finding anti-derivatives often referred to as **integration by substitution**.

(a) According to the chain rule, what is the derivative of $f(g(x))$?

$$\frac{d}{dx} f(g(x)) =$$

(b) Rewrite the chain rule as a rule for anti-differentiation.

(c) Make the substitution $u = g(x)$ and rewrite your above expression in terms of u.

(d) If you didn't already, replace $\frac{du}{dx}dx$ with du.

(e) Check that you have $\int f'(u)\, du = f(u) + C$.

So there it is. Let's put the technique to use.

Exercise 4.2.3. Consider $\int \exp(3x)\, dx$. Make the substitution $u = 3x$.

(a) Determine du. (Don't forget the dx!)

(b) Rewrite the anti-derivative in terms of u, find the anti-derivative, and then go back to x's.

Exercise 4.2.4. Use an appropriate substitution to determine the anti-derivatives below.

(a) $\int \sqrt{5x-2}\,dx$

(b) $\int \frac{4}{4x+7}\,dx$

(c) $\int \frac{2x}{1+x^2}\,dx$

(d) $\int \frac{3x^2-2}{x^3-2x+1}\,dx$

(e) $\int \cos(\exp(x)) \exp(x) \, dx$

(f) $\int \exp(\cos(x)) \sin(x) \, dx$

(g) $\int (6x-1)^{201} \, dx$

(h) $\int \tan(x) \, dx$

4.3 Integration by Parts

This worksheet discusses material corresponding roughly to section 4.2 of your textbook.

Recall the product rule for differentiation. Suppose u and v are functions of x. Then

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

Exercise 4.3.1. Rewrite the product rule for differentiation as a rule for anti-derivatives. You should have two $\int$'s on one side of the equation.

Exercise 4.3.2. Solve your previous equation for $\int u\frac{dv}{dx}\,dx$.

Exercise 4.3.3. Check that your equation is equivalent to the one below.

$$\int u\,dv = uv - \int v\,du$$

The equation $\int u\,dv = uv - \int v\,du$ is referred to as **integration by parts**. It gives us a technique for finding anti-derivatives as shown in the example below.

Exercise 4.3.4. Consider the anti-derivative $\int x\exp(x)\,dx$.

(a) Let $u = x$ and $dv = \exp(x)dx$. Observe that the original anti-derivative is of the form $\int u\,dv$.

(b) Determine du and v.

(c) Apply integration by parts. That is, plug u, v, du and dv into the equation $\int u\,dv = uv - \int v\,du$.

(d) Compute the remaining anti-derivative.

Exercise 4.3.5. Use integration by parts to determine the following anti-derivatives.

(a) $\int x \cos(x)\, dx$

(b) $\int 3x^2 \exp(x)\, dx$.

(c) $\int 2x \exp(-2x)\ dx$

(d) $\int \sqrt{x} \ln(x)\ dx$

4.4 Separable Differential Equations

This worksheet discusses material corresponding roughly to section 4.3 of your textbook.

Exercise 4.4.1. Consider the differential equation below.

$$\frac{dy}{dx} = \frac{x^2}{y}$$

(a) Rewrite the equation using differentials. That is, solve for dy.

(b) Use algebra to get everything with a y in it on one side of the equation, and everything with an x in it on the other side. (You should have $x^2\,dx$ on one side.)

(c) Find antiderivatives of each side. Make sure you remember to put a $+C$ on one side.

(d) Use implicit differentiation to check that your last equation provides a solution to the original differential equation.

A differential equation which can be written as $F(y)\,dy = G(x)\,dx$ for some functions F and G is called a **separable** differential equation.

Exercise 4.4.2. Show the differential equation below is separable and find all solutions. Don't forget your constant of integration. Check that your solution is indeed a solution by differentiating.

$$\exp(x)\frac{dy}{dx} = 1 + y^2$$

Exercise 4.4.3. Solve. Check that your solution is indeed a solution by differentiating.

$$(1 + x^2)\frac{dy}{dx} = 2xy$$

Exercise 4.4.4. Solve and check that your solution is correct.

$$\frac{dr}{dt} = \frac{2}{\sin(r) + t^2 \sin(r)}$$

Exercise 4.4.5. Solve and check that your solution is correct.

$$\frac{dy}{dx} = x \exp(x + y)$$

Exercise 4.4.6. Solve and check that your solution is correct.

$$\frac{dx}{dt} = \frac{t\sin(t)}{3x^2}$$

Exercise 4.4.7. Solve and check that your solution is correct.

$$\frac{dr}{dt} = \frac{t}{2r+1}$$

4.5 Separable Differential Equations

This worksheet discusses material corresponding roughly to section 4.3 of your textbook.

Exercise 4.5.1. Suppose that population P of a certain species grows in such a way that the instantaneous rate of change of the population $\frac{dP}{dt}$ (measured in population per year) is twice the current population level P.

(a) Write a differential equation which models the population dynamics. That is, take the information above and write down a differential equation which describes it.

(b) Solve the differential equation.

(c) Suppose that at time $t = 0$ years the population level is 20. Determine the population at time $t = 10$ years.

Exercise 4.5.2. Suppose a population P of a certain species grows in such a way that the instantaneous rate of change of the population (measured in population per year) is directly proportional to the current population level with unknown proportionality constant k.

(a) Write a differential equation which models the population dynamics.

(b) Solve the differential equation.

(c) Suppose that at time $t = 1$ years the population level is 10 and that at time $t = 2$ years the population level is 100. Determine a model which represents the population in year t for all t and use it to predict the population level in year 4.

Exercise 4.5.3. Let M be a constant. Determine constants A and B so that $\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$ as follows.

(a) Multiply the equation by $P(M-P)$.

(b) Plug in $P = M$ into the equation and solve for one of A or B.

(c) Go back and plug $P = 0$ into the equation and solve for the other of A or B.

(d) Plug in your values of A and B into the right-hand side of the original equation and combine the two fractions. Check that you get $\frac{1}{P(M-P)}$.

Exercise 4.5.4. Solve the separable differential equation below where k and M are constants.

$$\frac{dP}{dt} = kP(M-P)$$

Exercise 4.5.5. A population is growing according to the **logistic equation** given by the previous problem

$$\frac{dP}{dt} = kP(M - P)$$

where k and M are (positive) constants.

(a) Suppose at time $t = 0$ the population level is P_0. Determine the population level $P(t)$ at all times t.

(b) Assume that k, M, and P_0 are positive constants. Determine $\lim_{t \to \infty} P(t)$.

(c) Explain why M is referred to as the **carrying capacity** of the population.

4.6 Applications

This worksheet discusses material corresponding roughly to section 4.4 of your textbook.

Recall Newton's Second Law of Motion:

$$F = ma$$

This states that the total force (measured in Newtons) acting on an object is equal to its mass (in kilograms) times its acceleration (in meters per second per second). Also recall that acceleration $a = \frac{dv}{dt}$ and that velocity $v = \frac{dx}{dt}$, where x measures position.

Exercise 4.6.1. Suppose that the only force acting on an object with mass m is gravity, which is constant g.

(a) Use Newton's Second Law to determine an expression equal to $m\frac{dv}{dt}$.

(b) Determine the velocity of the object as a function of time.

(c) If the initial velocity is $v(0) = v_0$, determine the velocity of the object.

(d) Suppose the initial position is $x(0) = x_0$. Determine the position $x(t)$.

(e) What happens to the velocity as $t \to \infty$?

Before continuing with the physics, it will be worthwhile to do the following and make use of the result later.

Exercise 4.6.2. Let a and b be constants. Solve:

$$\frac{dy}{dx} = a + by$$

Exercise 4.6.3. Back to falling objects. Now suppose that the object in free fall experiences air resistance which is proportional to the velocity, and opposite the direction of motion.

(a) Given that gravity is acting to increase the velocity, and air resistance is acting to decrease velocity, use Newton's Second Law to determine an expression equal to $m\frac{dv}{dt}$. Call the constant of proportionality k.

(b) Determine the velocity of the object as a function of time.

(c) If the initial velocity is $v(0) = v_0$, determine the velocity of the object.

(d) Suppose the initial position is $x(0) = x_0$. Determine the position $x(t)$.

(e) What happens to the velocity as $t \to \infty$?

Exercise 4.6.4. Repeat the previous exercise assuming instead that air resistance is proportional to the velocity squared. *Hint*: At some point, you may want to find constants A and B so that $\frac{1}{\sqrt{\frac{gm}{k}} - v^2} = \frac{A}{\sqrt{\frac{gm}{k}} - v} + \frac{B}{\sqrt{\frac{gm}{k}} + v}$